# Generation of stochastic flows for stochastic PDE with non linear noise

Benjamin Gess
Max Planck Institute for Mathematics in the Sciences, Leipzig
& Universität Bielefeld

New Directions in Stochastic Analysis: Rough Paths, SPDEs and Related Topics
Zuse Institute Berlin (ZIB), March 2019

joint work with: Ben Fehrman
[G., Fehrman; arxiv, 2018].

We consider stochastic PDE of the type

$$\begin{aligned} \partial_t u &= \Delta(|u|^{m-1}u) + \nabla \cdot (A(x,u) \circ d\beta_t) \quad \text{on } \mathbb{T}^d \times (0,\infty), \\ u &= u_0 \quad \text{on } \mathbb{T}^d \times \{0\}, \end{aligned}$$

for $d \geq 1$, $m \in (0,\infty)$,

$$\nabla \cdot (A(x,u) \circ d\beta_t) = \sum_{i=1}^{d} \sum_{j=1}^{n} \partial_{x_i}(A^{i,j}(x,u) \circ d\beta_t^j).$$

Two motivating aspects:

1. Generation of stochastic flows?
2. Models/applications.

## Generation of stochastic flows by SPDE

- *Motivation*: Application of methods from dynamical systems to stochastic differential equations (multiplicative ergodic theorem, invariant manifolds, Lyapunov exponents).
- *Problem*: Do solutions to stochastic (partial) differential equations generate stochastic (semi-)flows?

- A map $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \times H \to H$ is a stochastic semi-flow if

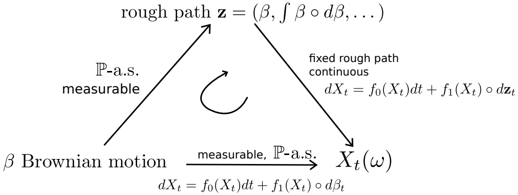
$$\varphi(t,s;\omega)x = \varphi(t,r;\omega)\varphi(r,s;\omega)x, \quad \forall s \le r \le t,\, \omega \in \Omega,\, x \in H. \tag{1}$$

- Consider SDE

$$\begin{aligned} dX_t^x(\omega) &= f_0(X_t^x(\omega))dt + f_1(X_t^x) \circ d\beta_t(\omega) \quad \text{on } (s,\infty) \\ X_s^x &= x \in H, \end{aligned} \tag{2}$$

  with $\beta$ a Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$.
- Obstacle: $X_t^x$ solves (2) for each $s$, $x$, $\mathbb{P}$-a.s.: There is a $\Omega_0 = \Omega_0(s,x) \subseteq \Omega$ with $\mathbb{P}[\Omega_0] = 1$ such that (2) it true for all $\omega \in \Omega_0$.
- Can only expect (1) for $\mathbb{P}$-a.e. $\omega \in \Omega$ (possibly depending on $s$, $x$).

- Finite dimensional SDE
  - Kolmogorov continuity theorem (e.g. [Kunita, 1980's])
  - Alternative: rough path theory [Lyons, 1998]




- Infinite dimensional SDE
  - Kolmogorov continuity theorem does not apply
  - Affine linear noise: transformation method (e.g. [Flandoli, 1995])
  - Semilinear SPDE

$$du = Lu\,dt + f(u, \nabla u)dt + H(x,u) \circ d\beta_t.$$

    Rough path theory [Gubinelli, Tindel; 2010], [Deya, Gubinelli, Tindel, 2012], [Diehl, Friz; 2012], [Hesse, Neamțu, 2018].
  - Stochastic conservation laws: [Lions, Perthame, Souganidis; 2014], [G., Friz; 2014], [Lions, Perthame, Souganidis; 2015], [G., Souganidis; 2015], [G., Souganidis; 2016], [Deya, Gubinelli, Hofmanova, Tindel, 2016].

**Open Problem:** Stochastic flow for

$$\partial_t u = \Delta u + \nabla \cdot (A(x,u) \circ d\beta_t).$$

## Applications

1. Introduction: Two leading aspects
   - Generation of stochastic flows by SPDE
   - Applications
2. Main results
3. Aspects of the proof

- Recall: We consider

$$\begin{aligned} \partial_t u &= \Delta(|u|^{m-1}u) + \nabla \cdot (A(x,u) \circ d\beta_t) && \text{on } \mathbb{T}^d \times (0,\infty), \\ u &= u_0 && \text{on } \mathbb{T}^d \times \{0\}, \end{aligned}$$

  for $d \geq 1$, $m \in (0,\infty)$.

- In particular

$$\partial_t u = \Delta u + \operatorname{div} f(x,u) + \nabla \cdot (A(x,u) \circ d\beta_t).$$

- Applications:
  - Limits of weakly interacting diffusions (mean field games)
  - Fluctuating hydrodynamics for zero range process
  - Dean-Kawasaki model (passive scalars in turbulent fluid with thermal noise)
  - Thin film equations
  - Geometric PDE

**Limits of weakly interacting diffusions:**

- Mean field interacting particles, for $i \in \{0, \dots, L\}$,

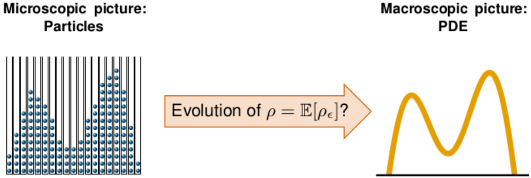
$$\mathrm{d}X_t^i = D^L(X_t^i, \frac{1}{L}\sum_{j\neq i}\delta_{X_t^j})\,\mathrm{d}W_t^i + \sigma^L(X_t^i, \frac{1}{L}\sum_{j\neq i}\delta_{X_t^j}) \circ \mathrm{d}\beta_t \;\text{ for }\; t \in (0,\infty),$$

  where $L \geq 1$, and $\beta_t$, $\{W_t^i\}_{i=1}^L$ are independent Brownian motions.
- Informally, from [Lasry, Lions; 2006], the density $m$ of the empirical law of the solution $X_t = (X_t^1, \dots, X_t^L)$ conditioned on $\beta$, in the mean field limit $L \to \infty$, satisfies

$$\begin{cases} \partial_t m = \frac{1}{2}\Delta\left(D^2(m)m\right) + \nabla\cdot(\sigma(x,m)m \circ \mathrm{d}\beta_t) & \text{in } \mathbb{T}^d \times (0,\infty), \\ m = m_0 & \text{on } \mathbb{T}^d \times \{0\}, \end{cases}$$

  provided the nonlocal nonlinearities $\sigma^L \to \sigma$ and $D^L \to D$ in approriate sense.

## Fluctuating hydrodynamics for zero range process:




- Hydrodynamic limit of a (symmetric) zero range particle process $\rho^\varepsilon \to \rho^0$ satisfies

$$\partial_t \rho^0 = \partial_{xx}\left(\Phi(\rho^0)\right) \quad \text{in } \mathbb{R}\times(0,\infty),$$

with $\Phi$ the mean local jump rate. E.g. $\Phi(\rho) = \rho\,|\rho|^{m-1}$.

- Fluctuations about hydrodynamic limit [Ferrari, Presutti, Vares; 1988]: Limit of $\frac{1}{\varepsilon}(\rho^\varepsilon - \rho^0) \to \rho^1$ satisfies

$$\partial_t \rho^1 = \partial_{xx}\left(\Phi'(\rho^0)\rho^1\right) + \partial_x\left(\sqrt{\Phi(\rho^0)}\,dW_t\right).$$

- Large deviations: [Dirr, Stamatakis, Zimmer; 2016]

$$\partial_t \rho = \partial_{xx}(\Phi(\rho)) + \partial_x\left(\sqrt{\varepsilon\Phi(\rho)}\,dW_t\right) \quad .$$

# Main results

1. Introduction: Two leading aspects
   - Generation of stochastic flows by SPDE
   - Applications
2. Main results
3. Aspects of the proof

- Recall

$$\begin{aligned} &\partial_t u = \Delta(|u|^{m-1}u) + \nabla\cdot(A(x,u)\circ d\mathbf{z}_t) && \text{on } \mathbb{T}^d\times(0,\infty),\\ &u = u_0 && \text{on } \mathbb{T}^d\times\{0\}, \end{aligned}$$

for $m\in(0,\infty)$.

- Obstacles
  - Irregularity: shocks, free interfaces
  - Non-uniqueness of weak solutions
- Assumptions:
  - Driving noise: For some $n\geq 1$, $\alpha\in(0,1)$,

$$z_t = (z_t^1,\ldots,z_t^n)\in C^{0,\alpha}\left([0,T];G^{\lfloor\frac{1}{\alpha}\rfloor}(\mathbb{R}^n)\right).$$

  - Regularity of the coefficients: For $\gamma>\frac{1}{\alpha}$,

$$\nabla_x A(x,v)\in C^{\gamma+2}(\mathbb{T}^d\times\mathbb{R}),\ \partial_v A(x,v)\in C^{\gamma+2}(\mathbb{T}^d\times\mathbb{R}).$$

  - No source:

$$\nabla_x\cdot A^t(x,0) = 0\in\mathbb{R}^n \text{ for each } x\in\mathbb{T}^d.$$

### Theorem

*Let* $u_0^1, u_0^2 \in L^2_+(\mathbb{T}^d)$ *and* $u^1$ *and* $u^2$ *be entropy solutions. Then*

$$\left\|u^1 - u^2\right\|_{L^\infty_t([0,\infty);L^1_x(\mathbb{T}^d))} \leq \left\|u_0^1 - u_0^2\right\|_{L^1_x(\mathbb{T}^d)}.$$

*In particular, entropy solutions are unique.*

### Theorem

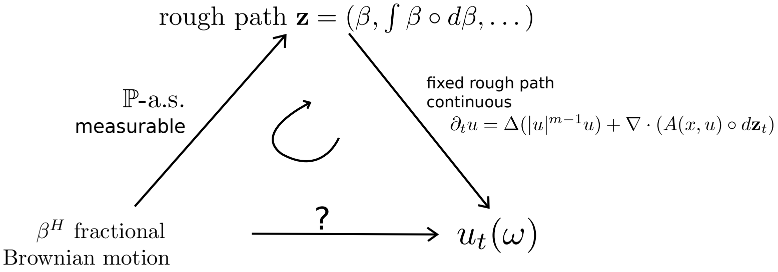
*Let* $u_0 \in L^2_+(\mathbb{T}^d)$. *There exists a unique non-negative entropy solution with initial data* $u_0$. *Furthermore,*

$$\|u\|_{L^\infty_t([0,\infty);L^1_x(\mathbb{T}^d))} \leq \|u_0\|_{L^1_x(\mathbb{T}^d)}.$$

Extensions:

- For $m > 2$ or $m = 1$, non-negativity of $u_0$ can be avoided.
- For $m \geq 3$ or $m = 1$, $u_0 \in (L^1 \cap L^2)(\mathbb{R}^d)$ the Cauchy problem can be treated by identical methods.
- Integrability: Localization allows extension to $L^1$-data.

Application to fractional Brownian motion:




### Theorem

*Let $t \in [0,\infty) \mapsto z_t(\omega)$ be the sample paths of a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, 1)$ on a probability space $\omega \in (\Omega, \mathscr{F}, \mathbb{P})$. Then $u$ defines a random dynamical system on $L^2_+(\mathbb{T}^d)$.*

### Theorem

*Let* $u_0 \in L^2_+(\mathbb{T}^d)$ *and* $T > 0$. *Let* $\{z^n\}_{n=1}^\infty, z \in C^{0,\alpha}\left([0,T]; G^{\lfloor \frac{1}{\alpha} \rfloor}(\mathbb{R}^n)\right)$ *satisfying*

$$\lim_{n\to\infty} d_\alpha(z^n, z) = 0.$$

*Let* $\{u^n\}_{n=1}^\infty$ *and* $u$ *be the pathwise kinetic solutions to driving signals* $\{z^n\}_{n=1}^\infty$ *and* $z$ *respectively. Then,*

$$\lim_{n\to\infty} \|u^n - u\|_{L^\infty([0,T];L^1(\mathbb{T}^d))} = 0.$$

# Aspects of the proof


1. Introduction: Two leading aspects
   - Generation of stochastic flows by SPDE
   - Applications
2. Main results
3. Aspects of the proof

Interlude: Kinetic formulation:

- Consider
$$\begin{cases} \partial_t u = \Delta(|u|^{m-1}u) & \text{on } \mathbb{T}^d \times (0,\infty), \\ u = u_0 & \text{on } \mathbb{T}^d \times \{0\}, \end{cases}$$
for $m \in (0,\infty)$.
- Kinetic formulation: Let
$$\chi(t,x,v) := 1_{v<u(t,x)} - 1_{v<0}.$$
Then
$$\begin{aligned} \partial_t \chi =& m\,\delta_{v=u}\mathrm{div}(|u|^{m-1}\nabla u) \\ =& m\,\mathrm{div}(\delta_{v=u}(|u|^{m-1}\nabla u)) - m\nabla_x(\delta_{v=u})(|u|^{m-1}\nabla u) \\ =& m\,\mathrm{div}(\delta_{v=u}(|v|^{m-1}\nabla u)) - m(\partial_u\delta_{v=u})|u|^{m-1}|\nabla u|^2 \\ =& m|v|^{m-1}\Delta\chi + \partial_v\left(m\delta_{v=u}|\frac{2}{m+1}\nabla|u|^{\frac{m+1}{2}}|^2\right) \\ =& m\,|v|^{m-1}\,\Delta_x\chi + \partial_v q \end{aligned}$$
for some non-negative measure $q$.
- Application: E.g. optimal regularity in Sobolev spaces [G., JEMS, 2019+].

- Consider

$$\begin{cases} \partial_t u = \Delta(|u|^{m-1}u) + \nabla\cdot(A(x,u)\circ d\mathbf{z}_t) & \text{on } \mathbb{T}^d\times(0,\infty), \\ u = u_0 & \text{on } \mathbb{T}^d\times\{0\}, \end{cases}$$

  for $m\in(0,\infty)$.
- Kinetic formulation: Let

$$\chi(t,x,v) := 1_{v<u(t,x)} - 1_{v<0}.$$

  Then

$$\partial_t\chi = m|v|^{m-1}\Delta_x\chi + \nabla_x\chi(\partial_v A(x,v)\circ d\mathbf{z}_t) - \partial_v\chi\left(\nabla_x\cdot A^t(x,v)\circ d\mathbf{z}_t\right) + \partial_v q$$

  for some non-negative measure $q$.
- Random test-functions (duality method) inspired by stochastic viscosity solutions.

- Recall: Kinetic formulation

$$\partial_t \chi = m|v|^{m-1}\Delta_x \chi + \nabla_x \chi(\partial_v A(x,v)\circ d\boldsymbol{z}_t) - \partial_v \chi\left(\nabla_x \cdot A^t(x,v)\circ d\boldsymbol{z}_t\right) + \partial_v q$$

- Consider, for each $t_0, t_1 \in [0,\infty)$ and $\rho_0 \in C_c^\infty(\mathbb{T}^d \times \mathbb{R})$,

$$\begin{aligned}
&\partial_t \rho_{t_0,t} = (\partial_v A(x,v)\circ d\boldsymbol{z}_t)\cdot \nabla_x \rho_{t_0,t} - (\nabla_x \cdot A^t(x,v)\circ d\boldsymbol{z}_t)\,\partial_v \rho_{t_0,t} \\
&\rho_{t_0,t_0} = \rho_0 .
\end{aligned}$$

- Then

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{T}^d} \chi(x,v,s)\rho_{t_0,s}(x,v)\,\mathrm{d}x dv\Big|_{s=t_0}^{t_1} \\
&= \int_{t_0}^{t_1}\int_{\mathbb{R}}\int_{\mathbb{T}^d} \left(m|v|^{m-1}\right)\chi(x,v,s)\Delta_x \rho_{t_0,s}(x,v)\,\mathrm{d}x dv\,\mathrm{d}s \qquad (3) \\
&- \int_{t_0}^{t_1}\int_{\mathbb{R}}\int_{\mathbb{T}^d} q(x,\xi,s)\partial_v \rho_{t_0,s}(x,v)\,\mathrm{d}x dv\,\mathrm{d}s.
\end{aligned}$$

- This gives a stable form of the SPDE: Say that $u$ is an entropy solution if $\chi$ satisfies (3).

- Recall

$$
\begin{aligned}
&\partial_t \rho_{t_0,t} = (\partial_v A(x,v) \circ d\mathbf{z}_t) \cdot \nabla_x \rho_{t_0,t} - (\nabla_x \cdot A^t(x,v) \circ d\mathbf{z}_t)\, \partial_v \rho_{t_0,t} \\
&\rho_{t_0,t_0} = \rho_0.
\end{aligned}
$$

- Characteristics:

$$
\begin{aligned}
&\mathrm{d}Y^{x,v}_{t_0,t} = \partial_v A(Y^{x,v}_{t_0,t}, \Pi^{x,v}_{t_0,t}) \circ d\mathbf{z}_{t_0,t} && \text{in } (0,t_0), \\
&\mathrm{d}\Pi^{x,v}_{t_0,t} = -\nabla_x \cdot A^t(Y^{x,v}_{t_0,t}, \Pi^{x,v}_{t_0,t}) \circ d\mathbf{z}_{t_0,t} && \text{in } (0,t_0), \\
&(Y^{x,v}_{t_0,0}, \Pi^{x,v}_{t_0,0}) = (x,v).
\end{aligned}
$$

- Solve the system of characteristics by rough path methods.
- Then

$$
\rho_{t_0,t}(x,v) = \rho_0 \left( Y^{x,v}_{t,t-t_0}, \Pi^{x,v}_{t,t-t_0} \right).
$$

- Note: spatially homogeneous case $(A(x,v) \equiv A(v))$ much simpler:

$$
\begin{aligned}
&Y^{x,v}_{t_0,t} = x + \partial_v A(v)\mathbf{z}_{t_0,t} && \text{in } (0,t_0), \\
&\Pi^{x,v}_{t_0,t} = v && \text{in } (0,t_0).
\end{aligned}
$$

## Uniqueness of entropy solutions

- Aim to estimate the $L^1$-difference

$$\int_{\mathbb{T}^d} |u^1 - u^2| \, \mathrm{d}x = \int_{\mathbb{R}} \int_{\mathbb{T}^d} |\chi^1 - \chi^2|^2 \, \mathrm{d}x dv = \int_{\mathbb{R}} \int_{\mathbb{T}^d} |\chi^1| + |\chi^2| - 2\chi^1 \chi^2 \, \mathrm{d}x dv$$
$$= \int_{\mathbb{R}} \int_{\mathbb{T}^d} \chi^1 \mathrm{sgn}(v) + \chi^2 \mathrm{sgn}(v) - 2\chi^1 \chi^2 \, \mathrm{d}x dv.$$

- Need to mollify on the right hand side:

$$\int_{\mathbb{T}^d} |u^1 - u^2| \, \mathrm{d}x = \lim_{\varepsilon,\delta \to 0} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \chi^{1,\varepsilon,\delta} \mathrm{sgn}^{\delta}(v) + \chi^{2,\varepsilon,\delta} \mathrm{sgn}^{\delta}(v) - 2\chi^{1,\varepsilon,\delta} \chi^{2,\varepsilon,\delta}.$$

- Leads to commutator errors when applying the equation.
- To control errors:
  - Exploit new cancellations
  - Use (new) regularity estimates on $u$.
- Spatially homogeneous case:

$$\partial_t u = \Delta(|u|^{m-1} u) + \nabla \cdot (A(u) \circ d\mathbf{z}_t).$$

Translation invariance yields $BV$-regularity of solutions (if $u_0 \in BV$).

- Essential new ingredient: Make use of full regularity

$$\int_0^T \int_{\mathbb{T}^d} |\nabla u^{\frac{m}{2}}|^2 dxdt < \infty$$

which corresponds to singular moment

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}} |v|^{-1} dq(t,x,v) < \infty.$$

- Note: True only for non-negative solutions.

**Existence of entropy solutions**

- New apriori estimates controlling

$$\|u\|_{L_t^{m+1} W_x^{\frac{2}{m+1},m+1}}.$$

K. Dareiotis and B. Gess.
Nonlinear diffusion equations with nonlinear gradient noise.
arXiv:1811.08356, 2018.

B. Fehrman and B. Gess.
Well-posedness of stochastic porous media equations with nonlinear, conservative noise.
arXiv:1712.05775, to appear in ARMA, 2019+.

B. Gess and P. E. Souganidis.
Scalar conservation laws with multiple rough fluxes.
Commun. Math. Sci., 13(6):1569–1597, 2015.

B. Gess and P. E. Souganidis.
Stochastic non-isotropic degenerate parabolic–hyperbolic equations.
Stochastic Process. Appl., 127(9):2961–3004, 2017.

*Happy Birthday, Terry!*