

Random dynamical systems for stochastic PDE with nonlinear noise

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Equadiff 2019
Leiden, July 2019

joint work with: Ben Fehrman
[G., Fehrman; ARMA, 2019].

We consider stochastic PDE of the type

$$\begin{aligned} \partial_t u &= \Delta(|u|^{m-1}u) + \nabla \cdot (A(x, u) \circ d\beta_t) && \text{on } \mathbb{T}^d \times (0, \infty), \\ u &= u_0 && \text{on } \mathbb{T}^d \times \{0\}, \end{aligned}$$

for $d \geq 1$, $m \in (0, \infty)$,

$$\nabla \cdot (A(x, u) \circ d\beta_t) = \sum_{i=1}^d \sum_{j=1}^n \partial_{x_i} (A^{i,j}(x, u) \circ d\beta_t^j).$$

Two motivating aspects:

- 1 Generation of stochastic flows?
- 2 Models/applications.

Generation of stochastic flows by SPDE

- *Motivation*: Application of methods from dynamical systems to stochastic differential equations (multiplicative ergodic theorem, invariant manifolds, Lyapunov exponents).
- *Problem*: Do solutions to stochastic (partial) differential equations generate stochastic (semi-)flows?

- A map $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \times H \rightarrow H$ is a stochastic semi-flow if

$$\varphi(t, s; \omega)x = \varphi(t, r; \omega)\varphi(r, s; \omega)x, \quad \forall s \leq r \leq t, \omega \in \Omega, x \in H. \quad (1)$$

- Consider SDE

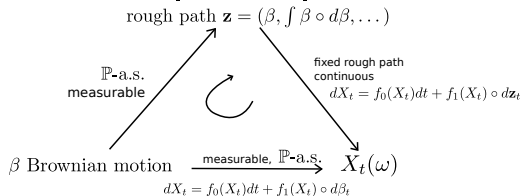
$$\begin{aligned} dX_t^x(\omega) &= f_0(X_t^x(\omega))dt + f_1(X_t^x) \circ d\beta_t(\omega) \quad \text{on } (s, \infty) \\ X_s^x &= x \in H, \end{aligned} \quad (2)$$

with β a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$.

- Obstacle: X_t^x solves (2) for each s, x , \mathbb{P} -a.s.: There is a $\Omega_0 = \Omega_0(s, x) \subseteq \Omega$ with $\mathbb{P}[\Omega_0] = 1$ such that (2) is true for all $\omega \in \Omega_0$.
- Can only expect (1) for \mathbb{P} -a.e. $\omega \in \Omega$ (possibly depending on s, x).

• Finite dimensional SDE

- Kolmogorov continuity theorem (e.g. [Kunita, 1980's])
- Alternative: rough path theory [Lyons, 1998]



• Infinite dimensional SDE

- Kolmogorov continuity theorem does not apply
- Affine linear noise: transformation method (e.g. [Flandoli, 1995])
- Semilinear SPDE

$$du = Lu dt + f(u, \nabla u)dt + H(x, u) \circ d\beta_t.$$

Rough path theory [Gubinelli, Tindel; 2010], [Deya, Gubinelli, Tindel, 2012], [Diehl, Friz; 2012], [Hesse, Neamțu, 2018].

- Stochastic conservation laws: [Lions, Perthame, Souganidis; 2014], [G., Friz; 2014], [Lions, Perthame, Souganidis; 2015], [G., Souganidis; 2015], [G., Souganidis; 2016], [Deya, Gubinelli, Hofmanova, Tindel, 2016].

Open Problem: Stochastic flow for

$$\partial_t u = \Delta u + \nabla \cdot (A(x, u) \circ d\beta_t).$$

Applications

- 1 Introduction: Two leading aspects
 - Generation of stochastic flows by SPDE
 - Applications
- 2 Main results
- 3 Aspects of the proof

- Recall: We consider

$$\begin{aligned} \partial_t u &= \Delta(|u|^{m-1}u) + \nabla \cdot (A(x, u) \circ d\beta_t) && \text{on } \mathbb{T}^d \times (0, \infty), \\ u &= u_0 && \text{on } \mathbb{T}^d \times \{0\}, \end{aligned}$$

for $d \geq 1$, $m \in (0, \infty)$.

- In particular

$$\partial_t u = \Delta u + \operatorname{div} f(x, u) + \nabla \cdot (A(x, u) \circ d\beta_t).$$

- Applications:

- Limits of weakly interacting diffusions (mean field games)
- Fluctuating hydrodynamics for zero range process
- Dean-Kawasaki model (passive scalars in turbulent fluid with thermal noise)
- Thin film equations
- Geometric PDE

Limits of weakly interacting diffusions:

- Mean field interacting particles, for $i \in \{0, \dots, L\}$,

$$dX_t^i = D^L(X_t^i, \frac{1}{L} \sum_{j \neq i} \delta_{X_t^j}) dW_t^i + \sigma^L(X_t^i, \frac{1}{L} \sum_{j \neq i} \delta_{X_t^j}) \circ d\beta_t \text{ for } t \in (0, \infty),$$

where $L \geq 1$, and $\beta_t, \{W_t^i\}_{i=1}^L$ are independent Brownian motions.

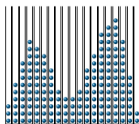
- Informally, from [Lasry, Lions; 2006], the density m of the empirical law of the solution $X_t = (X_t^1, \dots, X_t^L)$ conditioned on β , in the mean field limit $L \rightarrow \infty$, satisfies

$$\begin{cases} \partial_t m = \frac{1}{2} \Delta (D^2(m)m) + \nabla \cdot (\sigma(x, m)m \circ d\beta_t) & \text{in } \mathbb{T}^d \times (0, \infty), \\ m = m_0 & \text{on } \mathbb{T}^d \times \{0\}, \end{cases}$$

provided the nonlocal nonlinearities $\sigma^L \rightarrow \sigma$ and $D^L \rightarrow D$ in appropriate sense.

Fluctuating hydrodynamics for zero range process:

Microscopic picture:
Particles



Macroscopic picture:
PDE

Evolution of $\rho = \mathbb{E}[\rho_\varepsilon]$?



- Hydrodynamic limit of a (symmetric) zero range particle process $\rho^\varepsilon \rightarrow \rho^0$ satisfies

$$\partial_t \rho^0 = \partial_{xx} (\Phi(\rho^0)) \quad \text{in } \mathbb{R} \times (0, \infty),$$

with Φ the mean local jump rate. E.g. $\Phi(\rho) = \rho |\rho|^{m-1}$.

- Fluctuations about hydrodynamic limit [Ferrari, Presutti, Vares; 1988]: Limit of $\frac{1}{\varepsilon}(\rho^\varepsilon - \rho^0) \rightarrow \rho^1$ satisfies

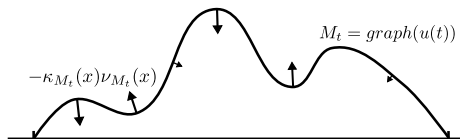
$$\partial_t \rho^1 = \partial_{xx} (\Phi'(\rho^0) \rho^1) + \partial_x (\sqrt{\Phi(\rho^0)} dW_t).$$

- Large deviations: [Dirr, Stamatakis, Zimmer; 2016]

$$\partial_t \rho = \partial_{xx} (\Phi(\rho)) + \partial_x \left(\sqrt{\varepsilon \Phi(\rho)} dW_t \right) \quad .$$

Geometric PDE

- Mean curvature flow:



- Stochastic mean curvature flow (graph case), e.g. [Lions, Souganidis, 1998],

$$\partial_t v = \sqrt{1 + |\nabla v|^2} \operatorname{div} \left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} \right) + \sqrt{1 + |\nabla v|^2} \circ dW.$$

- E.g. melting block of ice in water with thermal fluctuations
- In graph case in $1d$ (curve shortening flow) e.g. [Es-Sahir, Renesse 2012; Hofmanova, Röger, Renesse; 2015]

$$\partial_t u = \partial_x \left(\frac{\partial_x u}{1 + u^2} \right) + \sum_k \partial_x (\sqrt{1 + u^2} g^k(x) \circ d\beta^k).$$

Main results

- 1 Introduction: Two leading aspects
 - Generation of stochastic flows by SPDE
 - Applications
- 2 Main results
- 3 Aspects of the proof

- Recall

$$\begin{aligned} \partial_t u &= \Delta(|u|^{m-1}u) + \nabla \cdot (A(x, u) \circ dz_t) && \text{on } \mathbb{T}^d \times (0, \infty), \\ u &= u_0 && \text{on } \mathbb{T}^d \times \{0\}, \end{aligned}$$

for $m \in (0, \infty)$.

- Obstacles

- Irregularity: shocks, free interfaces
- Non-uniqueness of weak solutions

- Assumptions:

- Driving noise: For some $n \geq 1$, $\alpha \in (0, 1)$,

$$z_t = (z_t^1, \dots, z_t^n) \in C^{0, \alpha} \left([0, T]; G^{\lfloor \frac{1}{\alpha} \rfloor}(\mathbb{R}^n) \right).$$

- Regularity of the coefficients: For $\gamma > \frac{1}{\alpha}$,

$$\nabla_x A(x, v) \in C^{\gamma+2}(\mathbb{T}^d \times \mathbb{R}), \quad \partial_v A(x, v) \in C^{\gamma+2}(\mathbb{T}^d \times \mathbb{R}).$$

- No source:

$$\nabla_x \cdot A^t(x, 0) = 0 \in \mathbb{R}^n \text{ for each } x \in \mathbb{T}^d.$$

Theorem

Let $u_0^1, u_0^2 \in L^2_+(\mathbb{T}^d)$ and u^1 and u^2 be entropy solutions. Then

$$\|u^1 - u^2\|_{L^\infty_t([0, \infty); L^1_x(\mathbb{T}^d))} \leq \|u_0^1 - u_0^2\|_{L^1_x(\mathbb{T}^d)}.$$

In particular, entropy solutions are unique.

Theorem

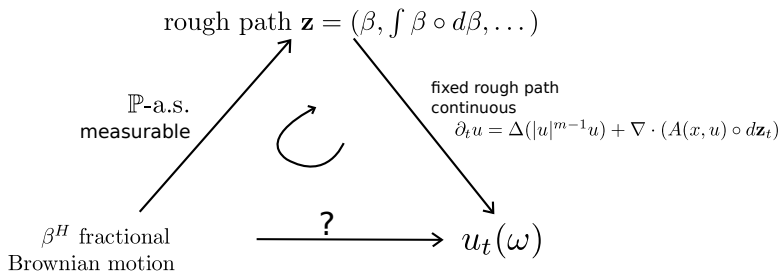
Let $u_0 \in L^2_+(\mathbb{T}^d)$. There exists a unique non-negative entropy solution with initial data u_0 . Furthermore,

$$\|u\|_{L^\infty_t([0, \infty); L^1_x(\mathbb{T}^d))} \leq \|u_0\|_{L^1_x(\mathbb{T}^d)}.$$

Extensions:

- For $m > 2$ or $m = 1$, non-negativity of u_0 can be avoided.
- For $m \geq 3$ or $m = 1$, $u_0 \in (L^1 \cap L^2)(\mathbb{R}^d)$ the Cauchy problem can be treated by identical methods.
- Integrability: Localization allows extension to L^1 -data.

Application to fractional Brownian motion:



Theorem

Let $t \in [0, \infty) \mapsto z_t(\omega)$ be the sample paths of a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, 1)$ on a probability space $\omega \in (\Omega, \mathcal{F}, \mathbb{P})$. Then u defines a random dynamical system on $L^2_+(\mathbb{T}^d)$.

Theorem

Let $u_0 \in L^2_+(\mathbb{T}^d)$ and $T > 0$. Let $\{z^n\}_{n=1}^\infty, z \in C^{0,\alpha}([0, T]; G^{\lfloor \frac{1}{\alpha} \rfloor}(\mathbb{R}^n))$ satisfying

$$\lim_{n \rightarrow \infty} d_\alpha(z^n, z) = 0.$$

Let $\{u^n\}_{n=1}^\infty$ and u be the pathwise kinetic solutions to driving signals $\{z^n\}_{n=1}^\infty$ and z respectively. Then,

$$\lim_{n \rightarrow \infty} \|u^n - u\|_{L^\infty([0, T]; L^1(\mathbb{T}^d))} = 0.$$

Aspects of the proof

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- Consider

$$\begin{cases} \partial_t u = \Delta(|u|^{m-1}u) + \nabla \cdot (A(x, u) \circ dz_t) & \text{on } \mathbb{T}^d \times (0, \infty), \\ u = u_0 & \text{on } \mathbb{T}^d \times \{0\}, \end{cases}$$

for $m \in (0, \infty)$.

- Kinetic formulation: Let

$$\chi(t, x, v) := 1_{v < u(t, x)} - 1_{v < 0}.$$

Then

$$\partial_t \chi = m|v|^{m-1} \Delta_x \chi + \nabla_x \chi (\partial_v A(x, v) \circ dz_t) - \partial_v \chi (\nabla_x \cdot A^t(x, v) \circ dz_t) + \partial_v q$$

for some non-negative measure q .

- Random test-functions (duality method) inspired by stochastic viscosity solutions.

- Essential new ingredient: Make use of full regularity

$$\int_0^T \int_{\mathbb{T}^d} |\nabla u^{\frac{m}{2}}|^2 dx dt < \infty$$

which corresponds to singular moment





$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}} |v|^{-1} dq(t, x, v) < \infty.$$

- Note: True only for non-negative solutions.

Existence of entropy solutions

- New a priori estimates controlling

$$\|u\|_{L_t^{m+1} W_x^{\frac{2}{m+1}, m+1}}.$$

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