Nonlinear stochastic partial differential equations in nonequilibrium statistical mechanics

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joint work with: Ben Fehrman.
[G., Fehrman; arxiv, 2019].

Introduction: Large deviations in zero range process

The zero range process

(could also consider simple exclusion, independent particles).

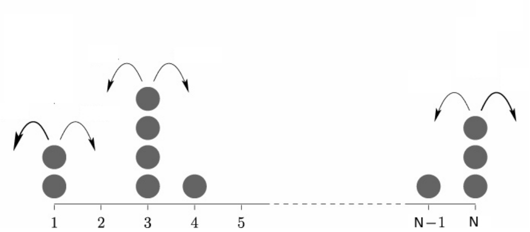

Figure: Harris, Rákos, Schütz; 2005

- Discrete d-dim. torus $\mathbb{T}_N^d := (\mathbb{Z}/(N\mathbb{Z}))^d$
- State space $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N^d}$, i.e. configurations $\eta : \mathbb{T}_N^d \to \mathbb{N}_0$: System in state η if container x contains $\eta(x)$ particles.
- Local jump rate function $g : \mathbb{N}_0 \to \mathbb{R}_0^+$, with $g(k) = 0$ iff $k = 0$, g Lipschitz continuous.

- Translation invariant, asymmetric, zero mean transition probability

$$p(x,y) = p(x-y), \quad \sum_k kp(k) = 0.$$

- Zero range process with jump rate g is the Markov jump process η_t on $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N^d}$ with generator

$$L_N f(\eta) = \sum_{x,y\in\mathbb{T}_N^d} (f(\eta^{x,y}) - f(\eta))g(\eta(x))p(x,y),$$

  where $f : \mathbb{M}_N \to \mathbb{R}$, $\eta^{x,y} = \eta - 1_{\{x\}} + 1_{\{y\}}$.
- Invariant probability measures ν_ρ (product structure and explicit), indexed by density $\rho \geq 0$.

- Hydrodynamic limit?

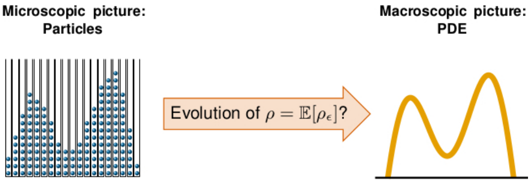

Figure: see Zimmer et. al.

- Empirical density field

$$\mu_t^N(x) := \frac{1}{N^d} \sum_k \delta_{\frac{k}{N}}(x) \eta_{tN^2}(k).$$

- Assume that $\mathscr{L}(\eta_0^N)$ is associated to a smooth profile ρ_0 with $0 < \rho_- \leq \rho_0 \leq \rho_+ < \infty$, that is, $\mathscr{L}(\eta_0^N)$ around Nx converges to $\nu_{\rho_0(x)}$ in probability.

Theorem (Hydrodynamic limit - Ferrari, Presutti, Vares; 1987)

For $\delta > 0$, G continuous, bounded,

$$\lim_{N\to\infty} \mathbb{P}\left[\left| < \mu_t^N, G > - < \rho_t \, dx, G > \right| > \delta \right] = 0$$

where $\rho(t,u)$ is the unique weak solution to

$$\partial_t \rho = \frac{1}{2} \partial_{xx} \Phi(\rho)$$
$$\rho(0) = \rho_0$$

with Φ the mean local jump rate

$$\Phi(\rho) = \mathbb{E}_{\nu_\rho}[g(\eta(0))].$$

E.g. $\Phi(\rho) = \rho \, |\rho|^{m-1}$.

- E.g. for constant intensity ($g(k) = 1_{k>0}$) get $\Phi(\rho) = \frac{\rho}{1+\rho}$.
- Independent particles, simple exclusion process: $\Phi(\rho) = \rho$.

- Let now ρ_0 constant.
- For $\mu \in D([0,t_0];\mathscr{M}_+)$, $\rho_t = d\mu_t/dx$ let

$$I_0(\mu) = \inf\left\{ \int_0^{t_0}\int |g|^2 dxds : g \in L^2_{t,x},\, \partial_t\rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)g) \right\}.$$

- E.g. independent particles: $\Phi(\rho) = \rho$.

Theorem (Large deviation principle, Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995)

For every measurable $A \subseteq D([0,t_0],\mathscr{M}_+)$,

$$\limsup_{N\to\infty} \frac{1}{N}\log\mathbb{P}[\mu^N \in \bar{A}] \le -\inf_{\mu\in\bar{A}} I_0(\mu) \le -\inf_{\mu\in A^\circ} I_0(\mu) \le \liminf_{N\to\infty}\frac{1}{N}\log\mathbb{P}[\mu^N \in A^\circ].$$

Informally, take $A = \{\rho\, dx\}$,

$$\mathbb{P}[\mu^N \approx \rho\, dx] \approx \exp\{-N\, I_0(\rho\, dx)\}.$$

Link to stochastic PDE

Link to stochastic PDE

- Large deviations [Dirr, Stamatakis, Zimmer; 2016] as special case of MFT [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015],

$$\partial_t \rho = \partial_{xx}(\Phi(\rho)) + \frac{1}{\sqrt{N}} \partial_x \left(\sqrt{\Phi(\rho)} dW_t \right).$$

- Model case: Dean-Kawasaki, independent particles, $\Phi(\rho) = \rho$, i.e.

$$\partial_t \rho = \partial_{xx} \rho + \frac{1}{\sqrt{N}} \partial_x (\sqrt{\rho} dW_t).$$

Informally same rate function as the zero range process:

- Informally applying the contraction principle to the solution map

$$F : \frac{1}{\sqrt{N}} dW \mapsto \rho$$

yields as a rate function

$$I(\rho) = \inf\{ I_{dW}(u) : F(u) = \rho \}.$$

- Schilder's theorem for Brownian sheet suggests

$$I_{dW}(u) = \int_0^T \int_{\mathbb{T}} |u|^2 \, dxdt.$$

- Get

$$I(\rho) = \inf\left\{ \int_0^T \int_{\mathbb{T}} |g|^2 \, dxdt : \partial_t \rho = \partial_{xx}(\Phi(\rho)) + \partial_x\left(\sqrt{\Phi(\rho)}g\right) \right\}.$$

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- Obstacle

$$\partial_t \rho = \partial_{xx}(\Phi(\rho)) + \frac{1}{\sqrt{N}} \partial_x \left(\sqrt{\Phi(\rho)} dW_t \right)$$

  1. not well-posed, supercritical -> no regularity structures
  2. Renormalization? Does renormalization appear in rate function? E.g. compare $\Phi^4_{2/3}$ [Hairer, Weber; 2014].

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- Ansatz: joint limit "small noise, ultraviolet cutoff"

$$\partial_t \rho^{N,K} = \partial_{xx}(\Phi(\rho^{N,K})) + \tfrac{1}{\sqrt{N}} \partial_x\left(\sqrt{\Phi(\rho^{N,K})} \circ dW_t^K\right),$$

  where $W^K = \sum_{k=1}^K e_k \beta^k$ is a spectral (smooth) approximation of $W = \sum_{k=1}^\infty e_k \beta^k$.

- Gives the correct rate function for $\frac{1}{N} << \frac{1}{K}$.

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Note: This is a particular case in which the link between *Macroscopic fluctuation theory* [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015] and *fluctuating hydrodynamics* [Landau-Lifshitz 1973, Spohn 1991] can be made rigorous.

Two ways to the LDP

- In the following concentrate on the case

$$\Phi(\rho) = \rho^m, \quad m \geq 1.$$

- We consider stochastic PDE of the type

$$\partial_t \rho^{N,K} = \Delta\left((\rho^{N,K})^m\right) + \frac{1}{\sqrt{N}} \mathrm{div}\left((\rho^{N,K})^{\frac{m}{2}} \circ dW_t^K\right), \qquad (*)$$

  on $\mathbb{T}^d \times (0,\infty)$, where $W^K = \sum_{k=1}^K e_k \beta^k$.
- Pathwise well-posedness of $(*)$: [Lions, Perthame, Souganidis; 2013], [Lions, Perthame, Souganidis; 2014], [G., Souganidis; 2014], [G., Souganidis; 2015], [G., Fehrman; 2017].

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Two ways to the LDP:

1. Γ-convergence of the rate functional: $N \uparrow \infty$ yields LDP for $(*)$ with rate function
$$I^K(\rho) = \inf\left\{\int_0^T \int_{\mathbb{T}} |g|^2\, dxdt : \partial_t \rho = \partial_{xx}(\Phi(\rho)) + \partial_x\left(\sqrt{\Phi(\rho)}P^K g\right)\right\}.$$
Then consider $K \uparrow \infty$.
2. Joint scaling: Weak convergence approach to LDP ($\frac{1}{N} << \frac{1}{K}$)

- Both approaches crucially depend on understanding the skeleton PDE.
- The skeleton equation

$$d\rho = \Delta\rho^m \, dt + \operatorname{div}\left(\rho^{\frac{m}{2}} g(t,x)\right) dt \qquad (*)$$

$$\rho(0,x) = \rho_0(x),$$

with $g \in L^2_{t,x}$?

- This leads to the key problem

Problem

1. *Existence and uniqueness of solutions to* $(*)$.
2. *Stability of solutions: Let* $g^n \rightharpoonup g$ *in* $L^2_{t,x}$ *with corresponding solutions* ρ^n, ρ. *Then*

$$\rho^n \to \rho$$

in $L^\infty_t L^1_x$.

- Difficulty: Stable a-priori bound? L^p framework does not work.
- Do we expect non-concentration of mass / well-posedness?

Scaling and criticality of the skeleton equation

- We consider
$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d$$
with $g \in L^q(\mathbb{R}_{+,t}; L^p(\mathbb{R}^d_x; \mathbb{R}^d_x))$ and $\rho_0 \in L^r(\mathbb{R}^d_x)$.
- Via rescaling (“zooming in”):
  - $p = q = 2$ is critical.
  - $r = 1$ is critical, $r > 1$ is supercritical.

Apriori-bounds and energy space

- Consider
$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d \qquad (*)$$
with $g \in L^2(\mathbb{R}_{+,t}; L^2(\mathbb{R}^d_x; \mathbb{R}^d_x))$.
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- L^1 estimate only gives
$$\int_{\mathbb{T}^d} \rho(t,x)dx = \int_{\mathbb{T}^d} \rho_0(x)dx.$$
- Use entropy-entropy dissipation: Evolution of entropy given by $\int_{\mathbb{T}^d} \log(\rho)\rho$. Informally gives
$$\int_{\mathbb{T}^d} \log(\rho)\rho \, dx\Big|_0^t + \int_0^t \int_{\mathbb{T}^d} (\nabla \rho^{\frac{m}{2}})^2 \lesssim \int_0^t \int_{\mathbb{T}^d} g^2.$$
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- Caution: Can only be true for non-negative solutions.
- Non-standard weak solutions, rewriting $(*)$ as
$$\partial_t \rho = 2\mathrm{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d$$
- Conclusion: Have to prove uniqueness within this class of solutions.

Ansatz for uniqueness: Show that every weak solution is a renormalized entropy solution.

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Let

$$\chi(t,x,\xi) = 1_{0<\xi<\rho(x,t)} - 1_{\rho(x,t)<\xi<0}.$$

Then, informally,

$$\partial_t \chi = m\xi^{m-1}\Delta_x \chi - g(x,t)(\partial_\xi \xi^{\frac{m}{2}})\nabla_x \chi + (\nabla_x g(x,t))\xi^{\frac{m}{2}}\partial_\xi \chi + \partial_\xi p$$

with p parabolic defect measure

$$p = \delta(\xi - \rho) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2.$$

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Or on "fluid level", informally, for F convex,

$$\partial_t F(\rho) - \Delta_x F^1(\rho) + g(x,t)\nabla_x F^2(\rho) + (\nabla_x g)(x,t)F^3(\rho) = -\int_\xi F''(\xi)p \le 0$$

where $(F^m)' = mf(\xi)\xi^{m-1}$, $(F^2)' = f(\xi)(\partial_\xi \xi^{\frac{m}{2}})$, $F^3 = f(\xi)\xi^{\frac{m}{2}}$.

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- Commutator errors,

$$\begin{aligned}
\partial_t \rho^\varepsilon = \varphi^\varepsilon * \partial_t \rho &= \varphi^\varepsilon * (\Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g)) \\
&= \Delta(\varphi^\varepsilon * \rho^m) + \mathrm{div}(\varphi^\varepsilon * (\rho^{\frac{m}{2}} g)) \\
&= \Delta(\rho^\varepsilon)^m + \mathrm{div}((\rho^\varepsilon)^{\frac{m}{2}} g) \\
&+ \Delta(\varphi^\varepsilon * \rho^m) - \Delta(\rho^\varepsilon)^m \\
&+ \mathrm{div}((\varphi^\varepsilon * \rho^{\frac{m}{2}}) g) - \mathrm{div}((\rho^\varepsilon)^{\frac{m}{2}} g) \\
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\end{aligned}$$

- Note: Additional commutator errors by commuting convolution and nonlinearities!
- Commutator estimate using non-standard (optimal) regularity $\rho^{\frac{m}{2}} \in L^2_t \dot{H}^1_x$
- Additional renormalization step to compensate low time integrability $\rho^{\frac{m}{2}} g \in L^1_t L^1_x$.

Uniqueness for renormalized entropy solutions (variable doubling)

- Additional errors from space-inhomogeneity (with little regularity)
- Note: Entropy dissipation measure

$$q(x,\xi,t) = \delta(\xi - \rho(x,t)) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2$$

  does not satisfy

$$\lim_{|\xi|\to\infty} \int_{t,x} q(x,\xi,t)\, dxdt = 0.$$

- Established arguments [Chen, Perthame; 2003] not applicable.

Theorem (The skeleton equation)

Let $g \in L^2([0,T]\times\mathbb{T}^d;\mathbb{R}^d)$*,* $\rho_0 \in L^1(\mathbb{T}^d)$ *non-negative and* $\int \rho_0 \log(\rho_0)dx < \infty$*,* $m \in [1,\infty)$*.*

1. *There is a unique weak solution*

$$\partial_t \rho = \Delta \rho^m + div(\rho^{\frac{m}{2}} g) \quad on\ \mathbb{R}_+ \times \mathbb{T}^d \tag{*}$$

For two weak solutions $\rho^1, \rho^2 \in L^\infty([0,T];L^1(\mathbb{T}^1))$ *we have*

$$\|\rho^1 - \rho^2\|_{L^\infty([0,T];L^1(\mathbb{T}^d))} \le \|\rho_0^1 - \rho_0^2\|_{L^1(\mathbb{T}^d)}.$$

2. *Let* $\{g_n\}_{n\in\mathbb{N}} \subseteq L^2([0,T]\times\mathbb{T}^d;\mathbb{R}^d)$ *with*

$$\lim_{n\to\infty} g_n = g \ \textit{weakly in}\ L^2([0,T]\times\mathbb{T}^d;\mathbb{R}^d)$$

and let $\rho_n \in L^1([0,T];L^1(\mathbb{T}^d))$ *be the corresponding solutions with control* g_n*. Then,*

$$\lim_{n\to\infty} \rho_n = \rho \ \textit{strongly in}\ L^1([0,T];L^1(\mathbb{T}^d))$$

where $\rho \in L^1([0,T];L^1(\mathbb{T}^d))$ *is the solution with control* g*.*

Consider

$$d\rho^N = \Delta(\rho^N)^m\, dt + \frac{1}{\sqrt{N}}\mathrm{div}\left(\Phi^{\frac{1}{2}}_{n(N)}(\rho^N)\circ dW^{K(N)}(t)\right).$$

Theorem (Large deviation principle)

Let $K(N), n(N) \to \infty$ with $\frac{K(N)^3}{N} \to 0$ for $N \to \infty$. For $\rho_0 \in L^{m+1}(\mathbb{T}^d)$ and $\rho \in L^\infty([0,T];L^1(\mathbb{T}^d))$ let

$$I_{\rho_0}(\rho) := \inf\left\{\frac{1}{2}\int_0^T \|g(s)\|^2_{L^2_x}\, ds : \; g \in L^2_{t,x}, \partial_t\rho = \Delta\rho^m + div(\rho^{\frac{m}{2}} g)\right\}.$$

Then,

1. *For all $\rho_0 \in L^{m+1}(\mathbb{T}^d)$, $\rho \mapsto I_{\rho_0}(\rho)$ is a good rate function on $L^\infty([0,T];L^1(\mathbb{T}^d))$.*
2. *The family $\{\rho^N\}$ satisfies the large deviation principle on $L^\infty([0,T];L^1(\mathbb{T}^d))$ with rate function I_{ρ_0}, uniformly on compact subsets of $L^{m+1}(\mathbb{T}^d)$.*

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