Large deviations for conservative, stochastic PDE and non-equilibrium fluctuations

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joint work with: Ben Fehrman.
[G., Fehrman; arxiv, 2020].

Introduction: Large deviations for the zero range process

1. Introduction: Large deviations for the zero range process
   - Fluctuations in the zero range process
   - Link to stochastic PDE

2. Two ways to the LDP
   - Scaling and criticality for the skeleton equation
   - Well-posedness of the skeleton equation

The zero range process

(could also consider simple exclusion, independent particles).

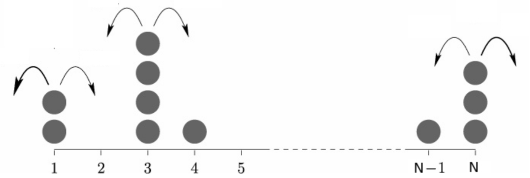

Figure: Harris, Rákos, Schütz; 2005

- State space $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N^d}$, i.e. configurations $\eta : \mathbb{T}_N^d \to \mathbb{N}_0$: System in state η if container x contains $\eta(x)$ particles.
- Local jump rate function $g : \mathbb{N}_0 \to \mathbb{R}_0^+$.
- Translation invariant, asymmetric, zero mean transition probability

$$p(x,y) = p(x-y), \quad \sum_k kp(k) = 0.$$

- Markov jump process η_t on $\mathbb{M}_N$.

- Hydrodynamic limit?

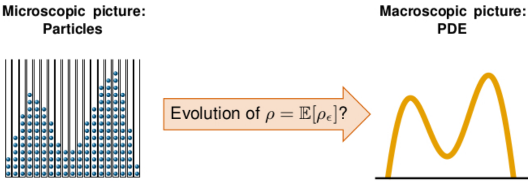

Figure: see Zimmer et. al.

- Empirical density field

$$\mu_t^N(x) := \frac{1}{N^d} \sum_k \delta_{\frac{k}{N}}(x) \eta_{tN^2}(k).$$

- [Hydrodynamic limit - Ferrari, Presutti, Vares; 1987]

$$\mu_t^N \rightharpoonup^* \bar{\rho}_t \, dx$$

with

$$\partial_t \bar{\rho} = \frac{1}{2} \partial_{xx} \Phi(\bar{\rho})$$

with Φ the mean local jump rate $\Phi(\rho) = \mathbb{E}_{\nu_\rho}[g(\eta(0))]$.

Rate of convergence?

- [Central limit fluctuations in non-equilibrium - Ferrari, Presutti, Vares; 1988]: Fluctuation density field

$$Y_t^N = \frac{1}{\sqrt{N}} \sum_k \delta_{\frac{k}{N}}(x)[\eta_{tN^2}(k) - <\eta_{tN^2}(k)>]$$

  for $t \geq 0$, $<\cdot>$ the expectation. Then,

$$\mathscr{L}(Y^N) \rightharpoonup^* \mathscr{L}(Y) \text{ for } N \to \infty$$

  with Y the solution to

$$dY_t = \partial_{xx}(\Phi'(\bar{\rho}_t(x))Y_t)\,dt + \partial_x(\sqrt{\Phi(\bar{\rho}_t(x))}dW_t)$$

  with dW space-time white noise.
- Therefore, expect

$$d(\mu_t^N, \bar{\rho}_t\,dx) \approx N^{-\frac{1}{2}}.$$

- [Large deviation principle, Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995]: Let now ρ_0 constant. Then, informally,

$$\mathbb{P}[\mu^N \approx \rho\, dx] \approx \exp\{-N\, I_0(\rho\, dx)\},$$

with rate function

$$I_0(\rho\, dx) = \inf\left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dxds : g \in L^2_{t,x}, \underbrace{\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho) g)}_{\text{"skeleton equation"}} \right\}.$$

Link to stochastic PDE

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Aim: Continuum model reflecting not only mean behavior but also fluctuations
Ansatz: Langevin dynamics

$$\partial_t \rho^N = \partial_{xx}\left(\Phi(\rho^N)\right) + \text{"fluctuations"}.$$

Aim:

$$d(\mu^N, \rho^N) << N^{-\frac{1}{2}}.$$

Concretely

$$\partial_t \rho^N = \partial_{xx}\left(\Phi(\rho^N)\right) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho^N)}dW_t\right). \qquad (\star)$$

Model case: Dean-Kawasaki, independent particles, $\Phi(\rho) = \rho$, i.e.

$$\partial_t \rho = \partial_{xx}\rho + \frac{1}{\sqrt{N}}\partial_x(\sqrt{\rho}dW_t).$$

Ansatz

$$\partial_t \rho^N = \partial_{xx}\left(\Phi(\rho^N)\right) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho^N)}dW_t\right). \qquad (\star)$$

Informal justification:

1. Physics: Fluctuation-dissipation relation, "fluctuating hydrodynamics"
2. Mean behavior / law of large numbers
$$\rho^N \to \bar{\rho} \quad \text{as } N \to \infty.$$
3. Central limit fluctuations: $Y^N := \sqrt{N}(\rho^N - \bar{\rho})$. Then, $\mathscr{L}(Y^N) \rightharpoonup^* \mathscr{L}(Y)$ with
$$\partial_t Y = \partial_{xx}\left(\Phi'(\bar{\rho})Y\right) + \partial_x\left(\sqrt{\Phi(\bar{\rho})}dW_t\right).$$
4. Large deviations: See below, large deviations of $(\star)$ are the same as for μ^N.

Informally, correct rare events:

- Informally applying the contraction principle to the solution map

$$F : \frac{1}{\sqrt{N}} dW \mapsto \rho$$

  yields as a rate function

$$I(\rho) = \inf\{I_{dW}(g) : F(g) = \rho\}.$$

- Schilder's theorem for Brownian sheet suggests

$$I_{dW}(g) = \int_0^T \int_{\mathbb{T}} |g|^2 \, dxdt.$$

- Get

$$I(\rho) = \inf\left\{ \int_0^T \int_{\mathbb{T}} |g|^2 \, dxdt : \partial_t \rho = \partial_{xx}(\Phi(\rho)) + \partial_x \left(\sqrt{\Phi(\rho)} g \right) \right\}.$$

- Obstacle
$$\partial_t\rho = \partial_{xx}(\Phi(\rho)) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho)}dW_t\right)$$
  1. not well-posed, supercritical -> no regularity structures
  2. Renormalization? Does renormalization appear in rate function? E.g. compare $\Phi^4_{2/3}$ [Hairer, Weber; 2014].
- Ansatz: joint limit "small noise, ultraviolet cutoff"

$$\partial_t\rho^{N,K} = \partial_{xx}\left(\Phi(\rho^{N,K})\right) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho^{N,K})}\circ dW_t^K\right)$$

where $W^K = \sum_{k=1}^K e_k\beta^k$ is a spectral (smooth) approximation of $W = \sum_{k=1}^\infty e_k\beta^k$.
- Gives the correct rate function for $\frac{1}{N} << \frac{1}{K}$.

Note: This is a particular case in which the link between *Macroscopic fluctuation theory* [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015] and *fluctuating hydrodynamics* [Landau-Lifshitz 1973, Spohn 1991] can be made rigorous.

Two ways to the LDP

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- In the following concentrate on the case
$$\Phi(\rho) = \rho^m, \quad m \geq 1.$$
- We consider stochastic PDE of the type
$$\partial_t \rho^{N,K} = \Delta\left((\rho^{N,K})^m\right) + \frac{1}{\sqrt{N}} \mathrm{div}\left((\rho^{N,K})^{\frac{m}{2}} \circ dW_t^K\right), \qquad (*)$$
on $\mathbb{T}^d \times (0,\infty)$, where $W^K = \sum_{k=1}^K e_k \beta^k$.
- Pathwise well-posedness of $(*)$: [Lions, Souganidis; 1998ff], [Lions, Perthame, Souganidis; 2013], [Lions, Perthame, Souganidis; 2014], [G., Souganidis; 2014], [G., Souganidis; 2015], [G., Fehrman; 2017], [Dareiotis, G.; 2019].

Two ways to the LDP:

1. Γ-convergence of the rate functional: $N \uparrow \infty$ yields LDP for $(*)$ with rate function
$$I^K(\rho) = \inf\left\{\int_0^T \int_{\mathbb{T}^d} |g|^2 \, dxdt : \partial_t \rho = \partial_{xx}\rho^m + \partial_x\left(\rho^{\frac{m}{2}} P^K g\right)\right\}.$$
Then consider $K \uparrow \infty$.
2. Joint scaling: Weak convergence approach to LDP ($\frac{1}{N} << \frac{1}{K}$).

What do we need to show?

- For example: First part of Γ-convergence of the rate function

$$I^K(\rho) = \inf\left\{ \int_0^T \int_{\mathbb{T}^d} |g|^2 \, dxdt : \partial_t \rho = \partial_{xx}\rho^m + \partial_x \left(\rho^{\frac{m}{2}} P^K g\right) \right\}.$$

- Let $\rho^K \to \rho$ need to show

$$I(\rho) \leq \liminf_K I^K(\rho^K).$$

- Assume we can choose g^K such that

$$I^K(\rho^K) = \int_0^T \int_{\mathbb{T}^d} |g^K|^2 \, dxdt \quad \text{and} \quad \partial_t \rho^K = \partial_{xx}\left(\rho^K\right)^m + \partial_x\left((\rho^K)^{\frac{m}{2}} P^K g^K\right).$$

- Then $g^K \rightharpoonup g$ in $L^2_{t,x}$. Need to show $\rho^K \to \rho$ with

$$\partial_t \rho = \partial_{xx}\rho^m + \partial_x\left(\rho^{\frac{m}{2}} g\right).$$

- Then

$$I(\rho) = \inf\left\{ \int_0^T \int_{\mathbb{T}^d} |g|^2 \, dxdt : \partial_t \rho = \partial_{xx}\rho^m + \partial_x \left(\rho^{\frac{m}{2}} g\right) \right\}$$
$$\leq \int_0^T \int_{\mathbb{T}^d} |g|^2 \, dxdt \leq \liminf_K \int_0^T \int_{\mathbb{T}^d} |g^K|^2 \, dxdt = \liminf_K I^K(\rho^K).$$

- Both approaches crucially depend on understanding the skeleton PDE.
- The skeleton equation

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}\left(\rho^{\frac{m}{2}} g(t,x)\right) \qquad (*)$$
$$\rho(0,x) = \rho_0(x),$$

with $g \in L^2_{t,x}$?

- This leads to the key problem

Problem

1. *Existence and uniqueness of solutions to* $(*)$.
2. *Stability of solutions: Let* $g^n \rightharpoonup g$ *in* $L^2_{t,x}$ *with corresponding solutions* ρ^n, ρ. *Then*
$$\rho^n \to \rho$$
in $L^\infty_t L^1_x$.

- Difficulty: Stable a-priori bound? L^p framework does not work.
- Do we expect non-concentration of mass / well-posedness?

Well-posedness of the skeleton equation

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Scaling and criticality of the skeleton equation

- We consider
$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d$$
with $g \in L^q(\mathbb{R}_{+,t}; L^p(\mathbb{R}^d_x; \mathbb{R}^d_x))$ and $\rho_0 \in L^r(\mathbb{R}^d_x)$.
- Via rescaling ("zooming in"):
  - $p = q = 2$ is critical.
  - $r = 1$ is critical, $r > 1$ is supercritical.

Apriori-bounds and energy space

- Consider
$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d \qquad (*)$$
with $g \in L^2(\mathbb{R}_{+,t}; L^2(\mathbb{R}^d_x; \mathbb{R}^d_x))$.
- L^1 estimate only gives
$$\int_{\mathbb{T}^d} \rho(t,x)dx = \int_{\mathbb{T}^d} \rho_0(x)dx.$$
- Use entropy-entropy dissipation: Evolution of entropy given by $\int_{\mathbb{T}^d} \log(\rho)\rho$. Informally gives
$$\int_{\mathbb{T}^d} \log(\rho)\rho \, dx\Big|_0^t + \int_0^t \int_{\mathbb{T}^d} (\nabla \rho^{\frac{m}{2}})^2 \lesssim \int_0^t \int_{\mathbb{T}^d} g^2.$$
- Caution: Can only be true for non-negative solutions.
- Non-standard weak solutions, rewriting $(*)$ as
$$\partial_t \rho = 2\mathrm{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d$$
- Conclusion: Have to prove uniqueness within this class of solutions.

Ansatz for uniqueness: Show that every weak solution is a renormalized entropy solution (extending the concepts of DiPerna-Lions, Ambrosio to nonlinear PDE). Let ρ be a weak solution to

$$\partial_t \rho = 2\mathrm{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d.$$

Let

$$\chi(t,x,\xi) = 1_{0<\xi<\rho(x,t)} - 1_{\rho(x,t)<\xi<0}.$$

Then, informally,

$$\partial_t \chi = m\xi^{m-1}\Delta_x \chi - g(x,t)(\partial_\xi \xi^{\frac{m}{2}})\nabla_x \chi + (\nabla_x g(x,t))\xi^{\frac{m}{2}}\partial_\xi \chi + \partial_\xi p$$

with p parabolic defect measure

$$p = \delta(\xi - \rho) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2.$$

Or on "fluid level", informally, for F convex,

$$\partial_t F(\rho) - \Delta_x F^1(\rho) + g(x,t)\nabla_x F^2(\rho) + (\nabla_x g)(x,t) F^3(\rho) = -\int_\xi F''(\xi) p \le 0$$

where $(F^m)' = mf(\xi)\xi^{m-1}$, $(F^2)' = f(\xi)(\partial_\xi \xi^{\frac{m}{2}})$, $F^3 = f(\xi)\xi^{\frac{m}{2}}$.

- How to make that rigorous? Take convolution
$$\rho^\varepsilon = \varphi^\varepsilon *_x \rho.$$
- Commutator errors,

$$\begin{aligned}
\partial_t \rho^\varepsilon = \varphi^\varepsilon * \partial_t \rho &= \varphi^\varepsilon * (\Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g)) \\
&= \Delta(\varphi^\varepsilon * \rho^m) + \mathrm{div}(\varphi^\varepsilon * (\rho^{\frac{m}{2}} g)) \\
&= \Delta(\rho^\varepsilon)^m + \mathrm{div}((\rho^\varepsilon)^{\frac{m}{2}} g) \\
&+ \Delta(\varphi^\varepsilon * \rho^m) - \Delta(\rho^\varepsilon)^m \\
&+ \mathrm{div}((\varphi^\varepsilon * \rho^{\frac{m}{2}}) g) - \mathrm{div}((\rho^\varepsilon)^{\frac{m}{2}} g) \\
&+ \mathrm{div}(\varphi^\varepsilon * (\rho^{\frac{m}{2}} g)) - \mathrm{div}((\varphi^\varepsilon * \rho^{\frac{m}{2}}) g).
\end{aligned}$$

- Note: Additional commutator errors by commuting convolution and nonlinearities!
- Commutator estimate using non-standard (optimal) regularity $\rho^{\frac{m}{2}} \in L^2_t \dot{H}^1_x$
- Additional renormalization step to compensate low time integrability $\rho^{\frac{m}{2}} g \in L^1_t L^1_x$.

Theorem

A function $\rho \in L_t^\infty L_x^1$ is a weak solution to

$$\partial_t \rho = 2div(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + div(\rho^{\frac{m}{2}} g)$$

if and only if ρ is a renormalized entropy solution.

Uniqueness for renormalized entropy solutions (variable doubling)

- Additional errors from space-inhomogeneity (with little regularity)
- Note: Entropy dissipation measure

$$q(x,\xi,t) = \delta(\xi - \rho(x,t)) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2$$

  does not satisfy

$$\lim_{|\xi|\to\infty} \int_{t,x} q(x,\xi,t)\, dxdt = 0.$$

- Established arguments [Chen, Perthame; 2003] not applicable.

Theorem (The skeleton equation)

Let $g \in L^2([0,T]\times\mathbb{T}^d;\mathbb{R}^d)$, $\rho_0 \in L^1(\mathbb{T}^d)$ *non-negative and* $\int \rho_0 \log(\rho_0) dx < \infty$, $m \in [1,\infty)$.

1. *There is a unique weak solution*

$$\partial_t \rho = \Delta \rho^m + div(\rho^{\frac{m}{2}} g) \quad on\ \mathbb{R}_+ \times \mathbb{T}^d. \qquad (*)$$

   For two weak solutions $\rho^1, \rho^2 \in L^\infty([0,T];L^1(\mathbb{T}^1))$ *we have*

$$\|\rho^1 - \rho^2\|_{L^\infty([0,T];L^1(\mathbb{T}^d))} \le \|\rho_0^1 - \rho_0^2\|_{L^1(\mathbb{T}^d)}.$$

2. *Let* $\{g_n\}_{n\in\mathbb{N}} \subseteq L^2([0,T]\times\mathbb{T}^d;\mathbb{R}^d)$ *with*

$$\lim_{n\to\infty} g_n = g \ \textit{ weakly in } \ L^2([0,T]\times\mathbb{T}^d;\mathbb{R}^d)$$

   and let $\rho_n \in L^1([0,T];L^1(\mathbb{T}^d))$ *be the corresponding solutions with control* g_n. *Then,*

$$\lim_{n\to\infty} \rho_n = \rho \ \textit{ strongly in } \ L^1([0,T];L^1(\mathbb{T}^d))$$

   where $\rho \in L^1([0,T];L^1(\mathbb{T}^d))$ *is the solution with control* g.

Consider

$$d\rho^N = \Delta(\rho^N)^m\,dt + \frac{1}{\sqrt{N}}\mathrm{div}\left(\Phi_{n(N)}^{\frac{1}{2}}(\rho^N)\circ dW^{K(N)}(t)\right).$$

Theorem (Large deviation principle)

Let $K(N), n(N) \to \infty$ with $\frac{K(N)^3}{N} \to 0$ for $N \to \infty$. For $\rho_0 \in L^{m+1}(\mathbb{T}^d)$ and $\rho \in L^\infty([0,T];L^1(\mathbb{T}^d))$ let

$$I_{\rho_0}(\rho) := \inf\left\{\frac{1}{2}\int_0^T \|g(s)\|^2_{L^2_x}ds : \ g \in L^2_{t,x}, \partial_t\rho = \Delta\rho^m + div(\rho^{\frac{m}{2}}g)\right\}.$$

Then, the family $\{\rho^N\}$ satisfies the large deviation principle on $L^\infty([0,T];L^1(\mathbb{T}^d))$ with good rate function I_{ρ_0}, uniformly on compact subsets of $L^{m+1}(\mathbb{T}^d)$.

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