

Conservative, stochastic PDE and non-equilibrium fluctuations

Benjamin Gess

Max Planck Institute for Mathematics in the Sciences, Leipzig
& Universität Bielefeld

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joint work with: Ben Fehrman, Nicolas Dirr.
[Fehrman, G.; arxiv, 2020], [Dirr, Fehrman, G.; arxiv, 2020].

- 1 Conservative SPDE as fluctuating continuum models
- 2 Two ways to the LDP, the skeleton equation

The zero range process

(could also consider simple exclusion, independent particles).

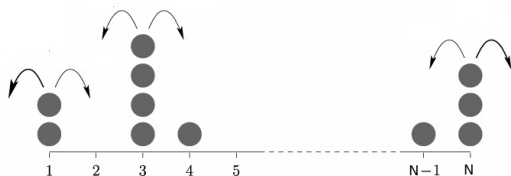


Figure: Harris, Rákos, Schütz; 2005

- State space $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N}$, i.e. configurations $\eta : \mathbb{T}_N \rightarrow \mathbb{N}_0$: System in state η if container k contains $\eta(k)$ particles.
- Local jump rate function $g : \mathbb{N}_0 \rightarrow \mathbb{R}_0^+$.
- Translation invariant, asymmetric, zero mean transition probability

$$p(k, l) = p(k - l), \quad \sum_k kp(k) = 0.$$

- Markov jump process $\eta(t)$ on \mathbb{M}_N .
- $\eta(k, t)$ = number of particles in box k at time t .

- Hydrodynamic limit? Multi-scale dynamics

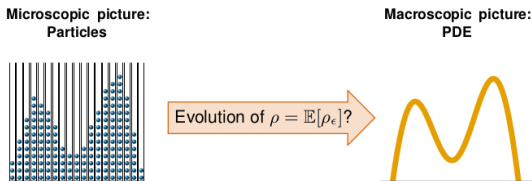


Figure: see Zimmer et. al.

- Empirical density field

$$\mu^N(x, t) := \frac{1}{N} \sum_k \delta_{\frac{k}{N}}(x) \eta(k, tN^2).$$

- [Hydrodynamic limit - Ferrari, Presutti, Vares; 1987]

$$\mu^N(t) \rightharpoonup^* \bar{\rho}(t) dx$$

with

$$\partial_t \bar{\rho} = \frac{1}{2} \partial_{xx} \Phi(\bar{\rho})$$

with Φ the mean local jump rate $\Phi(\rho) = \mathbb{E}_{v_\rho}[g(\eta(0))]$.

Rate of convergence?

- [Central limit fluctuations in non-equilibrium - Ferrari, Presutti, Vares; 1988]: Fluctuation density fields

$$\begin{aligned} Y^N(x, t) &= \frac{1}{\sqrt{N}} \sum_k \delta_{\frac{k}{N}}(x) [\eta(k, tN^2) - \mathbb{E}\eta(k, tN^2)] \\ &= \sqrt{N}(\mu^N(x, t) - \mathbb{E}\mu^N(x, t)) \end{aligned} \quad (*)$$

for $t \geq 0$. Then,

$$\mathcal{L}(Y^N) \rightharpoonup^* \mathcal{L}(Y) \text{ for } N \rightarrow \infty$$

with Y the solution to

$$dY(x, t) = \partial_{xx}(\Phi'(\bar{\rho}(x, t))Y(x, t)) dt + \partial_x(\sqrt{\Phi(\bar{\rho}(x, t))}dW(t))$$

with dW space-time white noise.

- Therefore, expect

$$d(\mu^N, \bar{\rho} dx) \approx N^{-\frac{1}{2}}.$$

- Note:

$$\mu^N(x, t) = \frac{1}{\sqrt{N}} Y^N(x, t) + \mathbb{E}\mu^N(x, t).$$

- Re-interpret (\star) as fluctuation correction

$$\begin{aligned}\mu^N(x, t) &= \frac{1}{\sqrt{N}} Y^N(x, t) + \mathbb{E}\mu^N(x, t) \\ &= \underbrace{\frac{1}{\sqrt{N}} Y(x, t) + \bar{\rho}(x, t)}_{:=\bar{\rho}^N(x, t)} + \underbrace{\frac{1}{\sqrt{N}} (Y^N - Y)(x, t)}_{=o(N^{-\frac{1}{2}})} + \underbrace{\mathbb{E}\mu^N(x, t) - \bar{\rho}(x, t)}_{=O(N^{-1})}.\end{aligned}$$

- Hence,

$$d(\mu^N, \bar{\rho}^N) = o(N^{-\frac{1}{2}}).$$

and notice that the *linearly* corrected continuum model $\bar{\rho}^N(x, t)$ satisfies

$$d\bar{\rho}^N(x, t) = \partial_{xx}(\Phi'(\bar{\rho}(x, t))\bar{\rho}^N(x, t)) dt + \frac{1}{\sqrt{N}} \partial_x(\sqrt{\Phi(\bar{\rho}(x, t))} dW(t)) \quad (\star)$$

i.e. a linear stochastic PDE with noise of small amplitude.

- Rare events? For (\star) we have rare events

$$\mathbb{P}[\bar{\rho}^N \approx \rho dx] \approx \exp\{-N \bar{I}(\rho dx)\},$$

with

$$\bar{I}(\rho dx) = \inf \left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dx ds : g \in L^2_{t,x}, \partial_t \rho = \partial_{xx}(\Phi'(\bar{\rho})\rho) + \partial_x(\Phi^{\frac{1}{2}}(\bar{\rho})g) \right\}.$$

- [Large deviation principle, Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995]: Let now ρ_0 constant. Then, informally,

$$\mathbb{P}[\mu^N \approx \rho dx] \approx \exp\{-N I(\rho dx)\},$$

with rate function

$$I(\rho dx) = \inf \left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dx ds : g \in L^2_{t,x}, \underbrace{\partial_t \rho = \partial_{xx} \Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)g)}_{\text{"skeleton equation"}} \right\}.$$

- Note: This does **not** coincide with the rate function of the linearly corrected continuum model $\bar{\rho}^N$,

$$\bar{I}(\rho dx) = \inf \left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dx ds : g \in L^2_{t,x}, \partial_t \rho = \partial_{xx}(\Phi'(\bar{\rho})\rho) + \partial_x(\Phi^{\frac{1}{2}}(\bar{\rho})g) \right\}.$$

- Ansatz: Derive a **nonlinear** fluctuating continuum model to simultaneously obtain higher order approximation and correct rare event behavior.

Ansatz: Langevin dynamics

$$\partial_t \rho^N = \partial_{xx} \left(\Phi(\rho^N) \right) + \frac{1}{\sqrt{N}} \partial_x \left(\sqrt{\Phi(\rho^N)} dW_t \right). \quad (\star)$$

Model case: Dean-Kawasaki, independent particles, $\Phi(\rho) = \rho$, i.e.

$$\partial_t \rho = \partial_{xx} \rho + \frac{1}{\sqrt{N}} \partial_x (\sqrt{\rho} dW_t).$$

Informal justification:

- 1 Physics: Fluctuation-dissipation relation, “fluctuating hydrodynamics”
- 2 Mean behavior / law of large numbers

$$\rho^N \rightarrow \bar{\rho} \quad \text{as } N \rightarrow \infty.$$

- 3 Central limit fluctuations: $Y^N := \sqrt{N}(\rho^N - \bar{\rho})$. Then, $\mathcal{L}(Y^N) \rightarrow^* \mathcal{L}(Y)$ with

$$\partial_t Y = \partial_{xx} (\Phi'(\bar{\rho}) Y) + \partial_x \left(\sqrt{\Phi(\bar{\rho})} dW_t \right).$$
- 4 Large deviations: See below, large deviations of (\star) are the same as for μ^N .

Informally, correct rare events:

- Informally applying the contraction principle to the solution map

$$F : \frac{1}{\sqrt{N}} dW \mapsto \rho$$

yields as a rate function

$$I(\rho) = \inf \{ I_{dW}(g) : F(g) = \rho \}.$$

- Schilder's theorem for Brownian sheet suggests

$$I_{dW}(g) = \int_0^T \int_{\mathbb{T}} |g|^2 dx dt.$$

- Get

$$I(\rho) = \inf \left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dx dt : \partial_t \rho = \partial_{xx} (\Phi(\rho)) + \partial_x \left(\sqrt{\Phi(\rho)} g \right) \right\}.$$

- Obstacle

$$\partial_t \rho = \partial_{xx} (\Phi(\rho)) + \frac{1}{\sqrt{N}} \partial_x \left(\sqrt{\Phi(\rho)} dW_t \right)$$

- ① not well-posed, supercritical \rightarrow no regularity structures
 - ② Renormalization? Does renormalization appear in rate function? E.g. compare $\Phi_{2/3}^4$ [Hairer, Weber; 2014].
- Decorrelation length of discrete system $= \frac{1}{N}$.
- Ansatz: joint limit “small noise, ultraviolet cutoff”

$$\partial_t \rho^{N,K} = \partial_{xx} \left(\Phi(\rho^{N,K}) \right) + \frac{1}{\sqrt{N}} \partial_x \left(\sqrt{\Phi(\rho^{N,K})} \circ dW_t^K \right)$$

where $W^K = \sum_{k=1}^K e_k \beta^k$ is a spectral (smooth) approximation of $W = \sum_{k=1}^{\infty} e_k \beta^k$.

- Gives the correct rate function for $\frac{1}{N} \ll \frac{1}{K}$.

Note: This is a particular case in which the link between *Macroscopic fluctuation theory* [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015] and *fluctuating hydrodynamics* [Landau-Lifshitz 1973, Spohn 1991] can be made rigorous.

Two ways to the LDP, the skeleton equation

- 1 Conservative SPDE as fluctuating continuum models
- 2 Two ways to the LDP, the skeleton equation

- In the following concentrate on the case

$$\Phi(\rho) = \rho^m, \quad m \geq 1.$$

- We consider stochastic PDE of the type

$$\partial_t \rho^{N,K} = \Delta \left((\rho^{N,K})^m \right) + \frac{1}{\sqrt{N}} \operatorname{div} \left((\rho^{N,K})^{\frac{m}{2}} \circ dW_t^K \right), \quad (*)$$

on $\mathbb{T}^d \times (0, \infty)$, where $W^K = \sum_{k=1}^K e_k \beta^k$.

- Pathwise well-posedness of (*): [Lions, Souganidis; 1998ff], [Lions, Perthame, Souganidis; 2013], [Lions, Perthame, Souganidis; 2014], [G., Souganidis; 2014], [G., Souganidis; 2015], [G., Fehrman; 2017], [Dareiotis, G.; 2019].

Two ways to the LDP:

- 1 Γ -convergence of the rate functional: $N \uparrow \infty$ yields LDP for (*) with rate function

$$I^K(\rho) = \inf \left\{ \int_0^T \int_{\mathbb{T}^d} |g|^2 dx dt : \partial_t \rho = \partial_{xx} \rho^m + \partial_x \left(\rho^{\frac{m}{2}} P^K g \right) \right\}.$$

Then consider $K \uparrow \infty$.

- 2 Joint scaling: Weak convergence approach to LDP ($\frac{1}{N} \ll \frac{1}{K}$).

- Both approaches crucially depend on understanding the skeleton PDE.
- The skeleton equation

$$\begin{aligned}\partial_t \rho &= \Delta \rho^m + \operatorname{div} \left(\rho^{\frac{m}{2}} g(t, x) \right) \\ \rho(0, x) &= \rho_0(x),\end{aligned}\tag{*}$$

with $g \in L^2_{t,x}$?

- This leads to the key problem

Problem

- 1 Existence and uniqueness of solutions to (*).
- 2 Stability of solutions: Let $g^n \rightharpoonup g$ in $L^2_{t,x}$ with corresponding solutions ρ^n, ρ .
Then

$$\rho^n \rightarrow \rho$$

in $L^\infty_t L^1_x$.

- Difficulty: Stable a-priori bound? L^p framework does not work.
- Do we expect non-concentration of mass / well-posedness?

Scaling and criticality of the skeleton equation

- We consider

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d$$

with $g \in L^q(\mathbb{R}_{+,t}; L^p(\mathbb{R}_x^d; \mathbb{R}_x^d))$ and $\rho_0 \in L^r(\mathbb{R}_x^d)$.

- Via rescaling (“zooming in”):
 - $p = q = 2$ is critical.
 - $r = 1$ is critical, $r > 1$ is supercritical.

Well-posedness and stability

- Obstacle: No L^p -estimates
- Optimal regularity: Entropy-entropy dissipation
- Uniqueness of weak solutions:
 - 1 Weak solutions are renormalized entropy solutions
New, nonlinear commutator errors. Exploit optimal regularity.
 - 2 Renormalized entropy solutions are unique
Lack of bounds on entropy dissipation measure: Refined proof of uniqueness.

Consider

$$d\rho^N = \Delta(\rho^N)^m dt + \frac{1}{\sqrt{N}} \operatorname{div} \left(\Phi_{n(N)}^{\frac{1}{2}}(\rho^N) \circ dW^{K(N)}(t) \right).$$

Theorem (Large deviation principle)

Let $K(N), n(N) \rightarrow \infty$ with $\frac{K(N)^3}{N} \rightarrow 0$ for $N \rightarrow \infty$. For $\rho_0 \in L^{m+1}(\mathbb{T}^d)$ and $\rho \in L^\infty([0, T]; L^1(\mathbb{T}^d))$ let

$$I(\rho) := \inf \left\{ \frac{1}{2} \int_0^T \|g(s)\|_{L_x^2}^2 ds : g \in L_{t,x}^2, \partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \right\}.$$

Then, the family $\{\rho^N\}$ satisfies the large deviation principle on $L^\infty([0, T]; L^1(\mathbb{T}^d))$ with good rate function I .



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