Stochastic scalar conservation laws

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joint work with: Panagiotis E. Souganidis, Benoit Perthame, Paul Gassiat
[G., Souganidis; CMS, 2014], [G., Souganidis; CPAM, 2016],
[G., Perthame, Souganidis; SINUM, 2016], [Gassiat, G.; ongoing].

Outline

Motivation

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- We will consider PDE driven by a 'rough' signal z of the type

$$du + \mathrm{div}(A(x,u) \circ dz) = 0.$$

If A is a diagonal matrix this becomes

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(x,u) \circ dz^j = 0$$

- For simplicity, often consider

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(u) \circ d\beta^j = 0.$$

- For example, stochastic Burgers' equation

$$du + \frac{1}{2}\partial_x u^2 \circ d\beta = 0.$$

Motivation

- The motivation comes from two directions: Relation to Hamilton-Jacobi equations, mean-field games.
- In the one-dimensional case: If v solves the Hamilton-Jacobi equation

$$dv + A(\partial_x v, x) \circ d\beta = 0$$

then $u = \partial_x v$ solves

$$du + \partial_x A(u, x) \circ d\beta = 0.$$

- But: The mathematical methods available for Hamilton-Jacobi equations (viscosity solutions) and scalar conservation laws (entropy solutions, kinetic methods) are very different.

Motivation

- Mean-field games going back to Lasry, Lions: Consider the SDE

$$dX_t^i = \sigma\left(X_t^i, \frac{1}{L-1}\sum_{j\neq i}\delta_{X_t^j}\right)\circ d\beta_t \quad \text{in } \mathbb{R}^N$$

  for $i = 1, \dots, L$.
- Then the empirical law of X converges to a measure π_t with density m_t which evolves according to

$$dm + \operatorname{div}(\sigma^*(x,m)\circ d\beta) = 0.$$

- Note that in general σ^* is not a diagonal matrix. We need the full generality of

$$du + \operatorname{div}(A(x,u)\circ d\beta) = 0.$$

Well-posedness

Well-posedness

Well-posedness

- Spatially inhomogeneous case:

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(x,u) \circ d\beta^j = 0. \tag{SSCL}$$

- If β is smooth, then (SSCL) makes sense classically

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(x,u) \dot{\beta}^j = 0.$$

- Aims:
  - Intrinsic solution: Define solutions to (SSCL) and prove well-posedness.
  - Consistency: Show that solutions to (SSCL) are obtained by approximation of the driving signal β.

Spatially homogeneous case

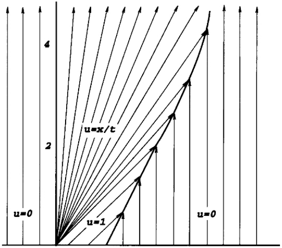
Reminder:

- Solutions to (deterministic) scalar conservation laws

$$du + \sum_j \partial_{x_j} A_j(u)\dot{\beta}^j = 0$$

  develop shocks (discontinuities).
- e.g. Burgers' equation

$$du + \frac{1}{2}\partial_x u^2 = 0$$
$$u(0) = 1_{[0,1]}$$



- At rarefractions, weak solutions are non-unique.
- Physically right solution is selected by entropy inequalities [Kružkov, 1970]

$$dS(u) + \sum_j \partial_{x_j} Q_j(u)\dot{\beta}^j \leq 0.$$

Spatially homogeneous case

- Alternative: kinetic solutions [Lions, Perthame, Tadmor; *JAMS*, 1994].
- For simplicity say $u_0 \geq 0$, which implies $u \geq 0$.
- We consider the characteristic function

$$\chi(t,x,\xi) := 1_{[0,u(t,x)]}(\xi).$$

Elementary calculation (if u were smooth, i.e. no shocks):

$$\begin{aligned}\partial_t \chi(t,x,\xi) :&= \delta_{\xi=u(t,x)}\partial_t u(t,x) = -\delta_{\xi=u(t,x)}\sum_j \partial_{x_j} A_j(u)\dot\beta^j \\ &= -\delta_{\xi=u(t,x)}\sum_j A_j'(u)\partial_{x_j} u\dot\beta^j = -\delta_{\xi=u(t,x)}\sum_j A_j'(\xi)\partial_{x_j} u\dot\beta^j \\ &= -\sum_j A_j'(\xi)\partial_{x_j} 1_{[0,u(t,x)]}(\xi)\dot\beta^j = -\sum_j A_j'(\xi)\partial_{x_j}\chi(t,x,\xi)\dot\beta^j.\end{aligned}$$

Spatially homogeneous case

- This is true up to shocks. The shocks introduce an error, the '*entropy dissipation measure*' m:

$$\partial_t \chi(t,x,\xi) + \sum_j A_j'(\xi)\partial_{x_j}\chi(t,x,\xi)\dot{\beta}^j = \partial_\xi m. \quad (1)$$

- In deterministic setting: u is an entropy solution iff $\chi(t,x,\xi) := 1_{[0,u(t,x)]}(\xi)$ is a kinetic solution to (1).
- Advantage: (1) is a linear equation in χ, at the expense of introducing the additional velocity variable ξ.
- In contrast to the non-linear situation, (1) can be transformed in a 'robust' form, i.e. in a form making sense also for non-smooth β.
- Here we follow the principle idea of stochastic viscosity solutions, i.e. do not transform the PDE itself, but put the transformation into test-functions.

Well-posedness

- Same calculations in the inhomogeneous case yield

$$d\chi+\sum_{j=1}^{N}(\partial_u A_j)(x,\xi)\partial_{x_j}\chi\dot{\beta}^j+\left(\sum_{j=1}^{N}\mathrm{div}_{x_j}A(x,\xi)\dot{\beta}^j\right)\partial_\xi\chi=\partial_\xi m.$$

- Test against solutions to

$$d\varphi-\sum_{j=1}^{N}(\partial_u A_j)(x,\xi)\partial_{x_j}\varphi\dot{\beta}^j-\left(\sum_{j=1}^{N}\mathrm{div}_{x_j}A(x,\xi)\dot{\beta}^j\right)\partial_\xi\varphi=0.$$

Then

$$d\int_{R^{N+1}}\chi\varphi\,dxd\xi=-\int_{R^{N+1}}\partial_\xi\varphi\,dm. \qquad (*)$$

- Roughly speaking: u is a pathwise entropy solution if χ satisfies $(*)$.

Theorem (Gess, Souganidis; CMS, 2015)

Pathwise entropy solutions are well-posed.

Long-time behavior

Long-time behavior

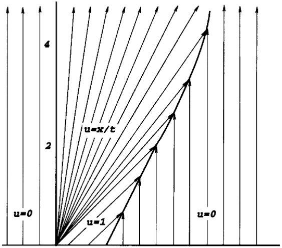
Stochastic Burgers' equation

Stochastic Burgers' equation - a simple example

Stochastic Burgers' equation

- e.g. Burgers' equation

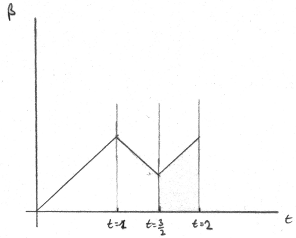
$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0$$
$$u(0) = 1_{[0,1]}$$



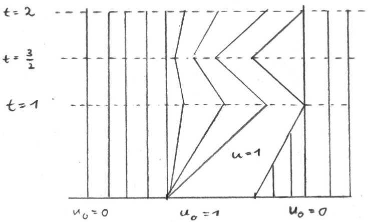
Stochastic Burgers' equation

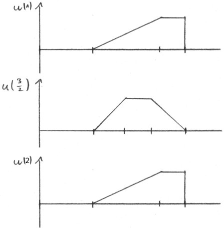
- Inhomogeneous case:

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ d\beta = 0$$
$$u(0) = 1_{[0,1]}$$



- Solution u:





Long-time behavior

Long-time behavior

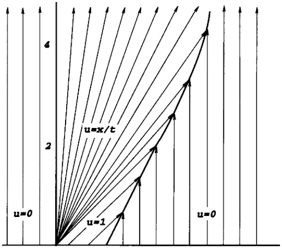
Long-time behavior

- We aim to analyze the long-time behavior of
$$\partial_t u + \operatorname{div} A(u) = 0$$
and
$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(u) \circ d\beta^j = 0$$
on the torus $\mathbb{T}^N$.
- We will show
$$u(t) \to \bar{u}_0 = \int_{\mathbb{T}^N} u_0(x)dx \quad \text{for } t \to \infty$$
in $L^1(\mathbb{T}^N)$.

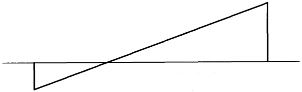
Some results from the deterministic case

- Recall: Burgers' equation

$$du + \frac{1}{2}\partial_x u^2 = 0$$
$$u(0) = 1_{[0,1]}$$



- For large times get asymptotic shape: "N-wave"



Some results from the deterministic case

Some existing results:

- $N=1$: [Lax; *CPAM*, 1957]. Rate for strictly convex flux:

$$\|u(t)-\bar{u}_0\|_\infty \lesssim t^{-1}.$$

- $N=2$: [Engquist, E; *CPAM*; 1993]. Rate for strictly convex flux:

$$\|u(t)-\bar{u}_0\|_\infty \lesssim t^{-1}.$$

- $N\geq 1$: [Chen, Frid; *ARMA*; 1999], [Chen, Perthame; *Proc. AMS*; 2009]. If A is 'genuinely nonlinear' then

$$\|u(t)-\bar{u}_0\|_1 \to 0 \quad \text{for } t\to\infty.$$

- Open problem: Rate of convergence for $N\geq 3$?

(New) rates for the deterministic case

- Assume that the flux A is *non-degenerate*: there exist $\theta \in (0,1]$ and $C > 0$ such that, for all $\sigma \in S^{N-1}$, $z \in \mathbb{R}$ and $\varepsilon > 0$,

$$|\{\xi \in \mathbb{R} : |A'(\xi) \cdot \sigma - z| \leq \varepsilon\}| \leq C\varepsilon^{\theta}.$$

  For example: For A strictly convex, $N = 1$ we have $\theta = 1$.

- Let u be the unique entropy solution to

$$\partial_t u + \sum_{j=1}^{N} \partial_{x_j} A_j(u) = 0.$$

Theorem (G., Souganidis; CPAM, 2016)

For $t \geq 1$ and $u_0 \in L^2(\mathbb{T}^N)$,

$$\|u(\cdot,t;\cdot,u_0) - \bar{u}_0\|_1 \leq t^{-\frac{\theta}{2+\theta}} \left(\|u_0\|_2^2 + 1\right).$$

(New) rates for the stochastic case

- Let us return to

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(u) \circ d\beta^j = 0.$$

- Again assume that A is genuinely nonlinear.

Theorem (G., Souganidis; CPAM, 2016)

For $t \geq 1$ and $u_0 \in L^2(\mathbb{T}^N)$,

$$\mathbb{E}\|u(\cdot,t;\cdot,u_0) - \bar{u}_0\|_1 \leq \left(\|u_0\|_2^2 + 1\right) t^{-\frac{\theta}{3+\theta}}.$$

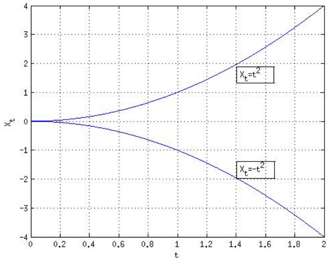
- E.g. $\theta = 1$: deterministic rate $t^{-\frac{1}{3}}$, stochastic rate $t^{-\frac{1}{4}}$. But: No claim of optimality.
- Note: Brownian motion scales like $\sqrt{t}$, which "slows down" characteristics.

Regularization by noise

Regularization by noise

Regularization by noise - linear case

- We recall: Consider
$$du + b(x)\cdot\nabla u = 0 \qquad \text{(TE)}$$
for non-Lipschitz b (but, say, Hölder continuous). E.g. $b(x) = \mathrm{sgn}(x)\sqrt{|x|}$.
- In general, characteristics collide causing shocks (i.e. discontinuities). Solution is not better than $u(t) \in BV$ even if u_0 is smooth.
- Weak solutions are non-unique: e.g. $b(x) = \mathrm{sgn}(x)\sqrt{|x|}$



- Question: Can noise restore uniqueness or increase regularity?

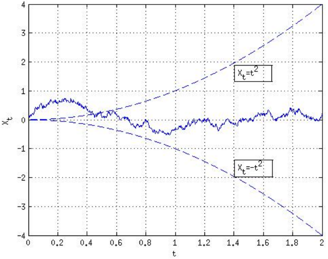
Regularization by noise - linear case

- Consider

$$du + b(x)\cdot\nabla u = -\nabla u \circ d\beta_t. \qquad \text{(STE)}$$

- Two (related) effects: regularization by noise, well-posedness by noise.
- Regularization by noise [Flandoli, Fedrizzi; *JFA*, 2013]: If u_0 is smooth then $u(t)$ is smooth.
- Well-posedness by noise [Flandoli, Gubinelli, Priola; *Invent. Math.*, 2010]: Weak solutions to (STE) are unique.

$$du + b(x)\cdot\nabla u = -\nabla u \circ d\beta_t$$
$$b(x) = \operatorname{sgn}(x)\sqrt{|x|}$$



- Entirely open: What about the nonlinear case, e.g. Burgers?

Quasi-solutions and averaging

- Consider the Burgers' equation

$$\partial_t u + \partial_x u^2 = 0 \quad \text{on } \mathbb{T} \tag{B}$$

- Weak solutions to (B) are not unique.
- We consider quasi-solutions [De Lellis, Otto, Westdickenberg; *ARMA*, 2003]: A weak solution u to (B) is a quasi-solution, if for some Radon measure m

$$\partial_t \chi + \xi \partial_x \chi = \partial_\xi m.$$

  Quasi-solutions to (B) are not unique.
- [De Lellis, Westdickenberg; *AIHP*, 2003]: There is a quasi-solution to (B) such that

$$u(t) \notin W^{\lambda,1} \qquad \text{for all } \lambda > \frac{1}{3}.$$

- Question: Does noise improve the situation?

(New) results for the stochastic case

- Consider the stochastic Burgers' equation

$$du + \partial_x u^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T} \tag{SB}$$

Theorem (G., Souganidis; CPAM, 2016)

Let u be a pathwise quasi-solution to (SB). Then, $t > 0$,

$$u(t) \in W^{\lambda,1} \quad \textit{for all } \lambda \in (0, \frac{4}{5}), \mathbb{P}\text{-a.s.}.$$

- Thus: quasi-solutions to (SB) are more regular than to (B), i.e. regularization by noise.

Numerics

Numerics

Numerics

- Semi-discretization scheme for

$$du + \sum_{j=1}^{N} \partial_{x_j} A^j(x,u) \circ dz^j = 0.$$

- As before set

$$\chi(t,x,\xi) := 1_{[0,u(t,x)]}(\xi)$$

which yields the kinetic form

$$d\chi + \sum_{j=1}^{N} \partial_\xi A^j(x,\xi) \partial_{x_j} \chi \circ dz^j + \sum_{j=1}^{N} \partial_{x_j} A^j(x,\xi) \partial_\xi \chi \circ dz^j = \partial_\xi m.$$

- Semi-discretization method via splitting and fast relaxation.

Numerics

- Given time steps $0 = t_0 < t_1 < \cdots < t_K = T$.
- Iteratively:
  - first solve the linear "free-streaming" transport equation

$$\partial_t \chi_{\Delta t} + \sum_{j=1}^{N} \partial_\xi A^j(x,\xi)\partial_{x_j}\chi_{\Delta t} \circ dz^j + \sum_{j=1}^{N} \partial_{x_j} A^j(x,\xi)\partial_\xi \chi_{\Delta t} \circ dz^j = 0$$

    on $[t_k, t_{k+1})$.
  - Then introduce a fast relaxation step, setting

$$u_{\Delta t}(t,x) := \int \chi_{\Delta t}(t-,x,\eta)d\eta$$

    and

$$\chi_{\Delta t}(t_{k+1},x,\xi) := 1_{[0,u_{\Delta t}(t_{k+1},x)]}(\xi).$$

- Finally, $u_{\Delta t}$ is taken as an approximation of the pathwise entropy solution.

Numerics

- Consider the spatially homogeneous case

$$du + \sum_{j=1}^{N} \partial_{x_j} A^j(u) \circ dz^j = 0.$$

- Convergence of this "transport collapse" scheme known in the deterministic case (i.e. $z_t = (t, \dots, t)$)
  - [Brenier, SIAM, 1984]: Via compactness methods and BV-estimates.
  - [Vasseur, SIAM, 1999]: Via compactness methods and averaging Lemma.
- Known proofs do *not* allow to obtain a rate of convergence.

Theorem (G., Perthame, Souganidis; SINUM, 2016)

Let $u_0 \in (BV \cap L^\infty)(\mathbb{R}^N)$. Then

$$\|u(t) - u_{\Delta t}(t)\|_{L^1} \le C\sqrt{\Delta z},$$

with a constant C given in terms of the data and

$$\Delta z := \max_{k=0,\dots,K-1} \sup_{t \in [t_k, t_{k+1}]} |z_t - z_{t_k}|.$$

Numerics

- Consider the spatially inhomogeneous case

$$du + \sum_{j=1}^{N} \partial_{x_j} A^j(x,u) \circ dz^j = 0.$$

- Established methods fail: Due to the spatial inhomogeneity
  - No *BV* estimates for u are available.
  - Averaging techniques do not apply.
- Convergence unknown even in the deterministic case.

Theorem (G., Perthame, Souganidis; SINUM, 2016)

Let $u_0 \in (L^1 \cap L^2)(\mathbb{R}^N)$. *Then, for* $\Delta t \to 0$,

$$u_{\Delta t} \to u \quad \textit{in } L^1([0,T] \times \mathbb{R}^N).$$

Thanks

Thanks!