Conservative SPDEs as fluctuating continuum models and non-equilibrium large deviations

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Content

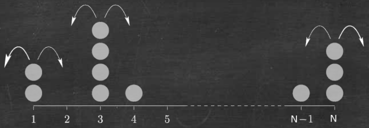
Conservative SPDEs as fluctuating continuum models

Large deviations in interacting particle systems

Parabolic-hyperbolic PDE with irregular drift

Conservative SPDE as fluctuating continuum models

The zero range process
(could also consider simple exclusion, independent particles).

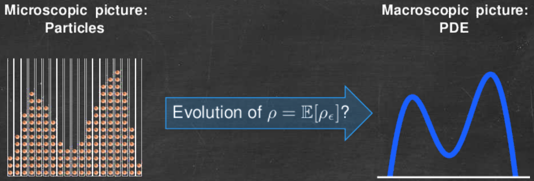


- State space $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N}$, i.e. configurations $\eta : \mathbb{T}_N \to \mathbb{N}_0$: System in state η if container k contains $\eta(k)$ particles.
- Local jump rate function $g : \mathbb{N}_0 \to \mathbb{R}_0^+$.
- Translation invariant, asymmetric, zero mean transition probability

$$p(k, l) = p(k - l), \quad \sum_k kp(k) = 0.$$

- Markov jump process $\eta(t)$ on $\mathbb{M}_N$.
- $\eta(k, t) = \text{number of particles in box } k \text{ at time } t.$

- **Hydrodynamic limit?** Multi-scale dynamics



- Empirical density field: $\mu^N(x,t) := \frac{1}{N}\sum_k \delta_{\frac{k}{N}}(x)\eta(k, tN^2)$.
- [Hydrodynamic limit - Ferrari, Presutti, Vares; 1987]

$$\mu^N(t) \rightharpoonup^* \bar{\rho}(t)\, dx$$

with

$$\partial_t \bar{\rho} = \partial_{xx}\Phi(\bar{\rho})$$

with Φ the mean local jump rate $\Phi(\rho) = \mathbb{E}_{\nu_\rho}[g(\eta(0))]$.
- Loss of information:
  - Error: $\mu^N = \bar{\rho} + o(1)$
  - Fluctuations, rare events?

Order of approximation

- Higher order expansion / fluctuation correction: Ansatz

$$\mu^N = \bar{\rho} + N^{-\frac{1}{2}} Y^{1,N} + N^{-1} Y^{2,N} + ...$$

What are $Y^{i,N}$?

- [Central limit fluctuations in non-equilibrium - Ferrari, Presutti, Vares; 1988]: Fluctuation density fields

$$Y^{1,N}(x,t) = N^{\frac{1}{2}}(\mu^N(x,t) - \mathbb{E}\mu^N(x,t)) \approx N^{\frac{1}{2}}(\mu^N(x,t) - \bar{\rho})$$

Then,

$$\mathcal{L}(Y^{1,N}) \rightharpoonup^* \mathcal{L}(Y^1) \text{ for } N \to \infty$$

with Y^1 the (Gaussian) solution to

$$dY^1 = \partial_{xx}(\Phi'(\bar{\rho})Y^1)\,dt + \partial_x(\Phi^{\frac{1}{2}}(\bar{\rho})\xi)$$

with ξ space-time white noise.

- Therefore,

$$\mu^N = \bar{\rho} + N^{-\frac{1}{2}} Y^1 + o(N^{-\frac{1}{2}}).$$

Ansatz: Langevin dynamics (nonlinear!)

$$\partial_t \rho^N = \partial_{xx}\left(\Phi(\rho^N)\right) + N^{-\frac{1}{2}}\partial_x\left(\Phi^{\frac{1}{2}}(\rho^N)\xi^N\right).$$

Model case: (correlated) Dean-Kawasaki, independent particles, $\Phi(\rho) = \rho$, i.e.

$$\partial_t \rho^N = \partial_{xx}\rho^N + N^{-\frac{1}{2}}\partial_x\left((\rho^N)^{\frac{1}{2}}\xi^N\right).$$

Informal justification:
1. Physics: Fluctuation-dissipation relation, "fluctuating hydrodynamics"
2. Law of large numbers, Central limit fluctuations (improved order of approximation) & *correct large deviations*

CLT for CSPDEs (done e.g. for SSEP)

$$\rho^N = \bar{\rho} + N^{-\frac{1}{2}}Y^1 + O(N^{-1}).$$

Conclude:

$$\mu^N = \rho^N + o(N^{-\frac{1}{2}}).$$

Expect even

$$\mu^N = \rho^N + O(N^{-1}).$$

Large deviations in interacting particle systems

(Im-)probability to observe a fluctuation ρ:

$$\mathbb{P}[\mu^N \approx \rho] = e^{-N\, I(\rho)} \quad N \text{ large}$$

Zero range process

$$I(\rho) = \|\partial_t \rho - \partial_{xx}\Phi(\rho)\|^2_{L^2_t H^{-1}_{\Phi(\rho)}}$$

$$= \inf\{ \underbrace{\|H\|^2_{L^2_t H^1_{\Phi(\rho)}}}_{=\int_{t,x} |\partial_x H|^2 \Phi(\rho)} : \underbrace{\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x H)}_{\text{"controlled nonlinear Fokker-Planck equation"}} \}.$$

Note: Informally coincides with the LDP expected for

$$\partial_t \rho = \partial_{xx}\Phi(\rho) + N^{-\frac{1}{2}} \partial_x(\Phi^{\frac{1}{2}}(\rho)\xi^N)$$

since

$$I(\rho) = \inf\{\|g\|^2_{L^2_{t,x}} : \overbrace{\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho) g)}^{\text{"skeleton equation"}}\}$$

$$= \inf\{\int_{t,x} |\underbrace{\Phi^{\frac{1}{2}}(\rho)\partial_x H}_{g}|^2 : \partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)\underbrace{\Phi^{\frac{1}{2}}(\rho)\partial_x H}_{g})\}.$$

What is rigorously known?

Theorem ([Large deviation principle, Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995])

For every open set $O \subseteq D([0,T],\mathcal{M}_+)$ we have

$$e^{-N \inf_{\mu \in O} \overline{I_{|A}}(\mu)} \lesssim \mathbb{P}[\mu^N \in O] \leq \mathbb{P}[\mu^N \in \bar{O}] \lesssim e^{-N \inf_{\mu \in \bar{O}} I(\mu)}$$

where A is the set of nice fluctuations $\mu = \rho\, dx$ with ρ a solution to

$$\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x H)$$

for some $\underline{H \in C^{1,3}_{t,x}}$.

This is a frequently observed problem: E.g. Fluctuations around Boltzmann equation [Rezakhanlou 1998], [Bodineau, Gallagher, Saint-Raymond, Simonella 2020]. Counter-examples [Heydecker; 2021].

Non-equilibrium thermodynamics: Macroscopic fluctuation theory [Öttinger 2005 "Beyond equilibrium thermodynamics"], [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim, 2015].

Large deviations lower bound (Kipnis, Donsker, Varadhan approach to rare events):

- Given a fluctuation ρ produced by a **regular** control $H \in C^{1,3}$

$$I(\rho) = \|H\|^2_{L^2_t H^1_{\Phi(\rho)}}, \quad \underbrace{\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x H)}_{\text{"controlled Fokker-Planck equation"}}.$$

- Observation: Controlled Fokker-Planck equation is the hydrodynamic limit of a weakly asymmetric particle system with asymmetry described by H.
- This corresponds to a change of measure on the microscopic level making ρ typical for $N \to \infty$
- Prove the hydrodynamic limit for the asymmetric system
- Problem: Only proves restricted lower bound, i.e. required regular control H.

Problem:

$$\text{upper rate function } I = \overline{I_{|A}} \text{ lower rate function ?}$$

Existence of a "recovery sequence"? Given fluctuation ρ so that $I(\rho dx) < \infty$, i.e. for some $H \in H^1_{\Phi(\rho)}$,

$$\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x \underbrace{H}_{\in H^1_{\Phi(\rho)}}).$$

Need to find sequence of nice fluctuations $\rho^\varepsilon \in A$ so that $\rho^\varepsilon \to \rho$ and $I(\rho^\varepsilon) \to I(\rho)$. That is, find $H^\varepsilon \in C^{1,3}([0,T] \times \mathbb{T})$ so that $\|H^\varepsilon\|_{L^2_t H^1_{\Phi(\rho)}} \to \|H\|_{L^2_t H^1_{\Phi(\rho)}}$ and

$$\partial_t \rho^\varepsilon = \partial_{xx}\Phi(\rho^\varepsilon) + \partial_x(\Phi(\rho^\varepsilon)\partial_x H^\varepsilon)$$

satisfies $\rho^\varepsilon \to \rho$.

Difficult problem: Open problem for the zero range process since [Benois, Kipnis, Landim; 1995].

Observation: Stability properties are better studied via the skeleton PDE

$$\begin{aligned}\partial_t\rho &= \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x H) \\ &= \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)\underbrace{\Phi^{\frac{1}{2}}(\rho)\partial_x H}_{L^2_{t,x}}) \\ &= \underbrace{\partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)\overbrace{g}^{\in L^2_{t,x}})}_{\text{"skeleton equation"}}\end{aligned}$$

Stability: $\quad g \mapsto \rho \quad$ continuous?

$\qquad L^2_{t,x} \to L^1_{t,x}$

I.e. Stability and uniqueness of a PDE with irregular coefficients $g \in L^2_{t,x}$.

How difficult is this?

- Difficulty: Stable a-priori bound? L^p framework does not work.
- Do we expect non-concentration of mass / well-posedness?

Parabolic-hyperbolic PDE with irregular drift

Skeleton equation

$$\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)\underbrace{g}_{\in L^2_{t,x}}).$$

Scaling and criticality of the skeleton equation

- We consider, $\Phi(\rho) = \rho^m$,

$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g)$$

  with $g \in L^q_t L^p_x$ and $\rho_0 \in L^r_x$.
- Via rescaling ("zooming in"):
  - $p = q = 2$ is critical.
  - $r = 1$ is critical, $r > 1$ is supercritical.

Recall: [Le Bris, Lions; CPDE 2008]

$$\partial_t \rho = \frac{1}{2} D^2 : (\sigma\sigma^* \rho) + \mathrm{div}(\rho g)$$

needs $g \in W^{1,1}_{loc,x}$, $\mathrm{div}\, g \in L^\infty$, $\sigma^* \nabla \rho \in L^2_{t,x}$.

Overview of ingredients of the proof:

- **Part 1:** Apriori-bounds; entropy-entropy dissipation estimates
- **Part 2:** Extending the concepts of DiPerna-Lions, Ambrosio, Le Bris-Lions to nonlinear PDE.
- **Part 3:** Uniqueness for renormalized entropy solutions (variable doubling): New treatment of kinetic dissipation measure. Exploit finite *singular* moments.

Part 1: Apriori-bounds

- Consider
$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d \tag{*}$$
with $g \in L^2_{t,x}$, $m \in [1,\infty)$. E.g.
$$\partial_t \rho = \Delta \rho + \mathrm{div}(\rho^{\frac{1}{2}} g).$$
- Use entropy-entropy dissipation: Evolution of entropy given by $\int_{\mathbb{T}^d} \log(\rho)\rho$. Informally gives
$$\int_x \log(\rho)\rho\,\Big|_0^t + \int_0^t \int_x (\nabla \rho^{\frac{m}{2}})^2 \lesssim \int_0^t \int_x g^2.$$
- Caution: Can only be true for non-negative solutions.
- Non-standard weak solutions, rewriting $(*)$ as
$$\partial_t \rho = 2\mathrm{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \mathrm{div}(\rho^{\frac{m}{2}} g).$$
- Stability in the control: for $g^\varepsilon \rightharpoonup g$ in $L^2_{t,x}$ by compactness $\rho^\varepsilon \to \hat{\rho}$ weak solution to $(*)$.
- Conclusion: Have to prove uniqueness within this class of solutions.

Part 2: Renormalization

Recall: The linear setting,

[DiPerna, Lions, Invent. Math. 1989; Ambrosio Invent. Math. 2004]

$$\partial_t \rho = \mathrm{div}(\rho g).$$

Then ρ is a renormalized solution, if for all smooth f we have

$$\partial_t f(\rho) = \mathrm{div}(f(\rho) g) - (f(\rho) + f'(\rho)\rho)\mathrm{div} g.$$

Let ρ be a weak solution to

$$\partial_t \rho = 2\mathrm{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d.$$

Show that every weak solution is a kinetic solution
(conjoining renormalization [DiPerna, Lions; Ambrosio] with kinetic solutions [Lions, Perthame, Tadmor, J. Amer. Math. Soc. 1994]).

Let

$$\chi(t,x,\xi) = f_\xi(\rho(x,t)) = 1_{0<\xi<\rho(x,t)} - 1_{\rho(x,t)<\xi<0}.$$

Then, informally,

$$\partial_t \chi = m\xi^{m-1}\Delta_x \chi - g(x,t)(\partial_\xi \xi^{\frac{m}{2}})\nabla_x \chi + (\nabla_x g(x,t))\xi^{\frac{m}{2}}\partial_\xi \chi + \partial_\xi q$$

with p parabolic defect measure

$$q = \delta(\xi - \rho(x,t))|\nabla \rho^{\frac{m+1}{2}}|^2.$$

- Note: Additional commutator errors by commuting convolution and nonlinearities.
- Commutator estimate using non-standard (optimal) regularity $\rho^{\frac{m}{2}} \in L_t^2 \dot{H}_x^1$
- Additional renormalization step to compensate low time integrability $\rho^{\frac{m}{2}} g \in L_t^1 L_x^1$.

Theorem

A function $\rho \in L_t^\infty L_x^1$ is a weak solution to

$$\partial_t \rho = 2div(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + div(\rho^{\frac{m}{2}} g)$$

if and only if ρ is a renormalized entropy solution (kinetic solution).

Part 3: Uniqueness for renormalized entropy solutions (variable doubling)

- Established arguments [Chen, Perthame; 2003] not applicable.
- Additional errors from space-inhomogeneity (with little regularity)
- Note: Entropy dissipation measure

$$q(x,\xi,t) = \delta(\xi - \rho(x,t))|\nabla \rho^{\frac{m+1}{2}}|^2 = \delta(\xi - \rho(x,t))\frac{\xi^m}{\xi^{m-1}}|\nabla \rho^{\frac{m}{2}}|^2$$

does not satisfy

$$\lim_{|\xi|\to\infty} \int_{t,x} q(x,\xi,t)\,dxdt = 0.$$

- Only finite singular moment

$$\int_{t,x,\xi} |\xi|^{-1} q(x,\xi,t)\,d\xi dxdt < \infty.$$

Theorem (The skeleton equation)

Let $g \in L^2_{t,x}$, ρ_0 non-negative and $\int \rho_0 \log(\rho_0) dx < \infty$. There is a unique weak solution to

$$\partial_t \rho = \Delta\Phi(\rho) + div(\Phi^{\frac{1}{2}}(\rho) g).$$

The map $g \mapsto \rho$ *is weak-strong continuous. E.g. including all*

$L^2_{t,x} \to L^1_{t,x}$

$\Phi(\rho) = \rho^m$, $m \in [1, \infty)$.

Theorem (LDP for zero range process)

The zero range process satisfies the full large deviations principle with rate function

$$I(\rho) = \|\partial_t \rho - \partial_{xx}\Phi(\rho)\|_{H^{-1}_{\Phi(\rho)}}.$$

References

B. Fehrman and B. Gess.
Non-equilibrium large deviations and parabolic-hyperbolic PDE with irregular drift.
arXiv:1910.11860 [math], Mar. 2022.

Construction of the recovery sequence ρ^n with smooth controls H^n: Recall

$$\begin{aligned}\partial_t \rho &= \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x H)\\ &= \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)\underbrace{\Phi^{\frac{1}{2}}(\rho)\partial_x H}_{g}).\end{aligned}$$

Let

$$g_n = g * \kappa^{\frac{1}{n}},\ \rho_{0,n} = ((\rho_0 \vee \frac{1}{n}) \wedge n) * \kappa^{\frac{1}{n}}$$

$$\psi_n = 0 \text{ on } [0, \frac{1}{2n}] \cup [2n, \infty) \ \ \psi_n = 1 \text{ on } [\frac{1}{n}, n]$$

Let ρ_n be the solution to

$$\partial_t \rho_n = \Delta\Phi(\rho_n) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho_n)\psi(\rho_n) g_n).$$

By comparison get

$$1/2n \leq \rho_n \leq 2n,$$

by parabolic regularity get ρ^n smooth. Consider the equation

$$\partial_t \rho_n = \Delta\Phi(\rho_n) - \nabla \cdot (\Phi(\rho_n)\nabla H_n)$$

as a (non-degenerate) elliptic equation for H_n. We get existence of a smooth H_n and thus ρ^n is a nice fluctuation.
The entropy dissipation estimate for ρ_n is still applicable since $\|\psi(\rho_n) g_n\|_{L^2} \leq \|g\|_{L^2}$. This allows to show convergence to a solution ρ, which by uniqueness is the pre-given function ρ.