Finite time extinction for stochastic sign fast diffusion and self-organized criticality.

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Outline

1. Self-organized criticality
2. Derivation of the BTW model from a cellular automaton
3. Finite time extinction and self-organized criticality
4. Finite time extinction for stochastic BTW

Self-organized criticality

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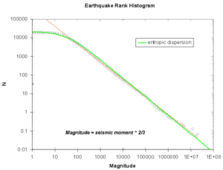
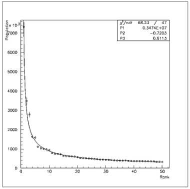
Self-organized criticality

- Many (complex) systems in nature exhibit power law scaling: The number of an event $N(s)$ scales with the event size s as

$$N(s) \sim s^{-\alpha}$$

- For example:

earthquakes	50 largest cities in the USA
	

Self-organized criticality

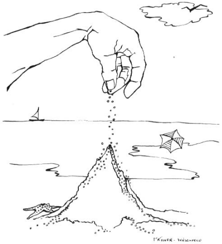
- Phase-transitions: The Ising model, ferromagnetism
- Critical temperature $T = T_c$:
  - strongly correlated: small perturbations can have global effects
  - no specific length scale (complex system, criticality)
- Observe: For $T = T_c$, power-law scaling for $N(s)$ being the number of $+1$ clusters of size s.

Self-organized criticality

- Ising model needs precise tuning $T = T_c$ to display power law scaling
- How can this occur in nature?
- Idea of self-organized criticality: [Bantay, Ianosi; Physica A, 1992]

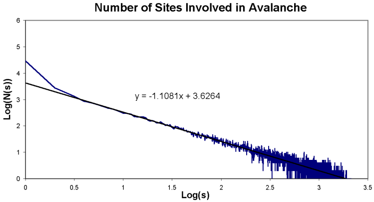
> *"Criticality" refers to the power-law behavior of the spatial and temporal distributions, characteristic of critical phenomena.*
> *"Self-organized" refers to the fact that these systems naturally evolve into a critical state without any tuning of the external parameters, i.e. the critical state is an attractor of the dynamics.*

- Bak, Tang, Wiesenfeld: Sandpile as a toy model of self-organized criticality



Sandpiles

- Two scales: Slow energy injection (adding sand), fast energy diffusion (avalanches)
- Criticality: No typical avalanche size, local perturbation may have global effects
- Power law scaling: $N(s)$ is the number of valances of size s.



Derivation of the BTW model from a cellular automaton

Derivation of the BTW model from a cellular automaton

Cellular automata model

- The following model goes back to [Bantay, Ianosi; Physica A, 1992].
- Aim: Define a cellular automaton displaying SOC.
- Consider an $N \times N$ square lattice, representing a discrete region $\mathscr{O} = \{(i,j)\}_{i,j=1}^{N}$.
- At each site (i,j) the height of the sandpile at time t is h_{ij}^{t}.
- The system is perturbed externally until the height h exceeds a threshold (critical) value h^c.

Cellular automata model

- Then, a toppling (avalanche) event occurs: The toppling at any 'activated' site (k,l) is described by:

$$h_{ij}^{t+1} \to h_{ij}^{t} - M_{ij}^{kl}, \quad \forall (i,j) \in \mathscr{O},$$

  where

$$M_{ij}^{kl} = \begin{cases} 4 & (k,l) = (i,j) \\ -1 & (k,l) \sim (i,j) \\ 0 & \text{otherwise.} \end{cases}$$

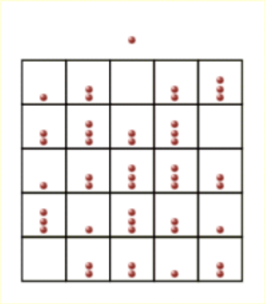
- Rewrite as:

$$h_{ij}^{t+1} - h_{ij}^{t} = -M_{ij}^{kl} H(h_{ij}^{t} - h_{ij}^{c}), \quad \forall (i,j) \in \mathscr{O},$$

  where H is the Heaviside function.
- The avalanches are continued until no site exceeds the threshold (which obviously happens after finitely many steps).

Cellular automata model

- As an example:



Continuum limit

- Passing to a continuum limit in

$$h_{ij}^{t+1} - h_{ij}^{t} = -M_{ij}^{kl} H(h_{ij}^{t} - h_{ij}^{c}), \quad \forall (i,j) \in \mathscr{O},$$

gives (informally)

$$\frac{\partial}{\partial t} X(t,\xi) = \Delta H(X(t,\xi) - X^c(\xi)),$$

where X is the continuous height-density function.

- In addition we impose zero Dirichlet boundary conditions:

$$H(X(t,\xi) - X^c(\xi)) = 0, \quad \text{on } \partial\mathscr{O}.$$

- Note: Only the relaxation/diffusion part modeled here. For full SOC-model we would have to include the external, random energy input.

Finite time extinction and SOC

Finite time extinction and self-organized criticality

Finite time extinction and SOC

- Question: Do avalanches end in finite time?
- Recall:

$$\frac{\partial}{\partial t}X(t,\xi) = \Delta H(X(t,\xi) - X^c(\xi)),$$

- We will restrict to the supercritical case, i.e. supposing $x_0 \geq X^c$.
- Substituting $X \to X - X^c$ and using $X \geq 0$ yields

$$\frac{\partial}{\partial t}X(t,\xi) = \Delta \mathrm{sgn}(X(t,\xi)),$$
$$X(0,\xi) = x_0(\xi)$$

with $x_0 \geq 0$ and zero Dirichlet boundary conditions:

$$\mathrm{sgn}(X(t,\xi)) = 0, \quad \text{on } \partial\mathscr{O}.$$

- Informally:

$$\Delta \mathrm{sgn}(X) = \delta_0(X)\Delta X + \mathrm{sgn}''(X)|\nabla X|^2.$$

- Avalanches end in finite time = Finite time extinction.

Finite time extinction for deterministic PDE

Finite time extinction for deterministic PDE

Finite time extinction for singular ODE

- Consider the singular ODE

$$\dot{f} = -cf^{\alpha}, \quad \alpha \in (0,1), \; c > 0.$$

- Then:

$$(f^{1-\alpha})' = -(1-\alpha).$$

- We obtain

$$f^{1-\alpha}(t) = f^{1-\alpha}(0) - (1-\alpha)ct$$

  which implies finite time extinction.

Finite time extinction and SOC

- [Diaz, Diaz; CPDE, 1979] finite time extinction (FTE) was first proven for

$$\frac{\partial}{\partial t}X(t,\xi) = \Delta \mathrm{sgn}(X(t,\xi)).$$

- In [Barbu; MMAS, 2012] another (more robust) approach based on energy methods was introduced.

Finite time extinction and SOC

- Informally the proof boils down to a combination of an L^1 and an L^∞ estimate of the solution:
- Informal L^∞ estimate:

$$\|X(t)\|_\infty \le \|x_0\|_\infty, \quad \forall t \ge 0.$$

- Informal L^1-estimate:

$$\begin{aligned}
\partial_t \int_{\mathscr{O}} |X(t,\xi)| d\xi &= \int_{\mathscr{O}} \operatorname{sgn}(X(t,\xi)) \Delta \operatorname{sgn}(X(t,\xi)) d\xi \\
&= -\int_{\mathscr{O}} |\nabla \operatorname{sgn}(X(t,\xi))|^2 d\xi \\
&\le -\left(\int_{\mathscr{O}} |\operatorname{sgn}(X(t,\xi))|^p d\xi \right)^{\frac{2}{p}} \\
&\le -(|\{\xi | X(t,\xi) \neq 0\}|)^{\frac{2}{p}},
\end{aligned}$$

for some (dimension dependent) $p > 2$. Note: $\frac{2}{p} < 1$.

Finite time extinction and SOC

- Observe

$$\int_{\mathcal{O}} |X(t,\xi)| d\xi \leq \|X(t)\|_\infty |\{\xi | X(t,\xi) \neq 0\}|.$$
$$\leq \|x_0\|_\infty |\{\xi | X(t,\xi) \neq 0\}|.$$

- Using this above gives

$$\partial_t \int_{\mathcal{O}} |X(t,\xi)| d\xi \leq -\frac{1}{\|x_0\|_\infty^{\frac{2}{p}}} \left(\int_{\mathcal{O}} |X(t,\xi)| d\xi \right)^{\frac{2}{p}}.$$

- We are left with the singular ODE

$$\dot{f} = -cf^\alpha, \quad \alpha \in (0,1), \; c > 0$$

for which we have seen that finite time extinction holds.

Finite time extinction for stochastic BTW

Finite time extinction for stochastic BTW

The stochastic BTW model

- In [Díaz-Guilera; EPL (Europhysics Letters), 1994], [Giacometti, Diaz-Guilera; Phys. Rev. E, 1998], [Díaz-Guilera; Phys. Rev. A, 1992] it was pointed out that it is more realistic to include stochastic perturbations.
- This leads to SPDE of the form

$$dX_t = \Delta H(X_t - X^c) + B(X_t - X^c)dW_t,$$

  with appropriate diffusion coefficients B.
- We study linear multiplicative noise, i.e.

$$dX_t = \Delta H(X_t - X^c) + \sum_{k=1}^{N} f_k(X_t - X^c)d\beta_t^k.$$

- Question: Do avalanches end in finite time?

The stochastic BTW model

- Recall:

$$dX_t = \Delta \mathrm{sgn}(X_t) + \sum_{k=1}^{N} f_k X_t \, d\beta_t^k,$$

with zero Dirichlet boundary conditions.

- Finite time extinction can be reformulated in terms of the extinction time

$$\tau_0(\omega) := \inf\{t \geq 0 | X_t(\omega) = 0, \text{ a.e. in } \mathscr{O}\}.$$

We distinguish the following concepts:

(F1) Extinction with positive probability for small initial conditions: $\mathbb{P}[\tau_0 < \infty] > 0$, for small $X_0 = x_0$.

(F2) Extinction with positive probability: $\mathbb{P}[\tau_0 < \infty] > 0$, for all $X_0 = x_0$.

(F3) Finite time extinction: $\mathbb{P}[\tau_0 < \infty] = 1$, for all $X_0 = x_0$.

Some known results

- Existence and uniqueness of solutions to

$$dX_t \quad \in \quad \Delta\phi(X_t)dt + \sum_{k=1}^{N} f_k X_t d\beta_t^k$$

  with ϕ being possibly multi-valued goes back to [Barbu, Da Prato, Röckner; CMP, 2009].
- In the same paper (F1) for the Zhang model is shown for $d = 1$.
- In [Barbu, Da Prato, Röckner; JMAA, 2012] this was extended to prove (F1) for the BTW model for $d = 1$.
- In the recent work [Röckner, Wang; JLMS, 2013] finite time extinction for the Zhang model has been solved.
- In case of additive noise

$$dX_t \quad \in \quad \Delta \mathrm{sgn}(X_t)dt + dW_t,$$

  ergodicity has been shown for $d = 1$ in [Gess, Tölle; JMPA, to appear].
- In [Barbu, Röckner; ARMA, 2013] (F1) has been shown for the related stochastic total variation flow for $d \leq 3$.

Main result

Theorem (Main result)

Let $x_0 \in L^\infty(\mathscr{O})$, X *be the unique variational solution to BTW and let*

$$\tau_0(\omega) := \inf\{t \geq 0 | X_t(\omega) = 0, \text{ for a.e. } \xi \in \mathscr{O}\}.$$

Then finite time extinction holds, i.e.

$$\mathbb{P}[\tau_0 < \infty] = 1.$$

For every $p > \frac{d}{2} \vee 1$, *the extinction time* $\tau_0(\omega)$ *may be chosen uniformly for* x_0 *bounded in* $L^p(\mathscr{O})$.

Transformation

- Recall:

$$dX_t = \Delta \mathrm{sgn}(X_t) + \sum_{k=1}^{N} f_k X_t d\beta_t^k,$$

- Our approach to FTE will be based on considering the following transformation: Set $\mu_t := \sum_{k=1}^N f_k \beta_t^k$, $\tilde{\mu} := \sum_{k=1}^N f_k^2$ and $Y_t := e^{-\mu_t} X_t$. An informal calculation shows

$$\partial Y_t \in e^{\mu_t} \Delta \mathrm{sgn}(Y_t) - \tilde{\mu} Y_t. \qquad (*)$$

- Compare the deterministic setting:

$$\partial Y_t \in \Delta \mathrm{sgn}(Y_t).$$

Outline of the proof

- There are two main ingredients of the proof:
  1. A uniform control on $\|X_t\|_p$ for all $p \geq 1$.
  2. An energy inequality for a weighted L^1-norm.
- On an intuitive level the arguments become clear by approximating

$$r^{[m]} := |r|^{m-1} r \to \mathrm{sgn}, \quad \text{for } m \downarrow 0.$$

  To make these arguments rigorous, in fact a different (non-singular, non-degenerate) approximation of sgn is used.
- In the following let Y_t be a solution to

$$\partial_t Y_t \in e^{\mu_t} \Delta Y_t^{[m]} - \tilde{\mu} Y_t.$$

Step 1: Informal L^p bound

- **Step 1:** A uniform control on $\|X_t\|_p$ for all $p \geq 1$.
- We may informally compute for all $p \geq 1$:

$$\begin{aligned}\partial_t \int_{\mathscr{O}} |Y_t|^p d\xi =& p \int_{\mathscr{O}} Y_t^{[p-1]} e^{\mu_t} \Delta Y_t^{[m]} d\xi \\ =& -\frac{4(p-1)mp}{(p+m-1)^2} \int_{\mathscr{O}} e^{\mu_t} \left(\nabla |Y_t|^{\frac{p+m-1}{2}}\right)^2 d\xi \\ & + \frac{pm}{p+m-1} \int_{\mathscr{O}} |Y_t|^{p+m-1} \Delta e^{\mu_t} d\xi.\end{aligned}$$

- Taking $p > 1$ and then $m \to 0$ we may "deduce" from this

$$\partial_t \int_{\mathscr{O}} |Y_t|^p d\xi \leq 0.$$

Step 2: Informal "L^1" bound

- **Step 2:** An energy inequality for a weighted L^1-norm.

$$\partial_t \int_{\mathscr{O}} |Y_t|^p d\xi = -\frac{4(p-1)mp}{(p+m-1)^2} \int_{\mathscr{O}} e^{\mu_t} \left(\nabla |Y_t|^{\frac{p+m-1}{2}}\right)^2 d\xi + \frac{pm}{p+m-1} \int_{\mathscr{O}} |Y_t|^{p+m-1} \Delta e^{\mu_t} d\xi, \; p \geq 1.$$

- Choose $p = m+1$ and let $m \to 0$. We obtain

$$\partial_t \int_{\mathscr{O}} |Y_t| d\xi = -\int_{\mathscr{O}} e^{\mu_t} (\nabla \mathrm{sgn}(Y_t))^2 d\xi + \frac{1}{2} \int_{\mathscr{O}} \Delta e^{\mu_t} d\xi$$

- Recall: deterministic case

$$\partial_t \int_{\mathscr{O}} |Y_t| d\xi = -\int_{\mathscr{O}} |\nabla \mathrm{sgn}(Y_t)|^2 d\xi.$$

Step 2: Informal "L^1" bound

Key trick: Use a weighted L^1-norm

- Let φ be the classical solution to

$$\Delta\varphi = -1, \quad \text{on } \mathcal{O}$$
$$\varphi = 1, \quad \text{on } \partial\mathcal{O}.$$

  Note $1 \le \varphi \le \|\varphi\|_\infty =: C_\varphi$.

- We informally compute

$$\partial_t \int_{\mathcal{O}} \varphi |Y_t| d\xi = -\int_{\mathcal{O}} \varphi e^{\mu_t} (\nabla \mathrm{sgn}(Y_t))^2 d\xi + \frac{1}{2}\int_{\mathcal{O}} \Delta(\varphi e^{\mu_t}) d\xi.$$

- Note

$$\Delta(\varphi e^{\mu_t}) = -e^{\mu_t} + 2\nabla\varphi \cdot \nabla e^{\mu_t} + \varphi \Delta e^{\mu_t}$$

  has a negative sign for small times ($e^{\mu_t} \approx 1$)!

- Shift the initial time

$$\partial_t \int_{\mathcal{O}} e^{-\mu_s} \varphi |Y_t| d\xi = -\int_{\mathcal{O}} e^{\mu_t - \mu_s} \varphi (\nabla \mathrm{sgn}(Y_t))^2 d\xi + \frac{1}{2}\int_{\mathcal{O}} \mathrm{sgn}(Y_t)^2 \Delta e^{\mu_t - \mu_s} \varphi d\xi$$

Thanks

Thanks!