Finite time extinction for stochastic sign fast diffusion and self-organized criticality.

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Outline

Derivation of the BTW model from a cellular automaton

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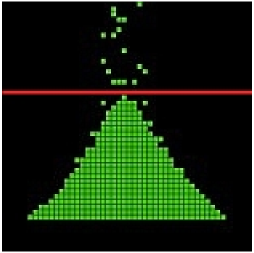
Cellular automata model

- The following model goes back to [Bantay, Ianosi; Physica A, 1992]. Here we follow the exposition in [Barbu; MMAS, 2012].
- Cellular automata model for a sandpile:



Cellular automata model

- Consider an $N \times N$ square lattice, representing a discrete region $\mathscr{O} = \{(i,j)\}_{i,j=1}^{N}$.
- At each site (i,j) the height of the sandpile at time t is h_{ij}^{t}.
- The system is perturbed externally until the height h exceeds a threshold (critical) value h^{c}.
- Then, a toppling (avalanche) event occurs: The toppling at any 'activated' site (k,l) is described by:

$$h_{ij}^{t+1} \to h_{ij}^{t} - M_{ij}^{kl}, \quad \forall (i,j) \in \mathscr{O},$$

  where

$$M_{ij}^{kl} = \begin{cases} 4 & (k,l) = (i,j) \\ -1 & (k,l) \sim (i,j) \\ 0 & \text{otherwise.} \end{cases}$$

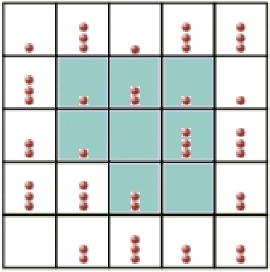
- Rewrite as:

$$h_{ij}^{t+1} - h_{ij}^{t} = -M_{ij}^{kl} H(h_{ij}^{t} - h_{ij}^{c}), \quad \forall (i,j) \in \mathscr{O},$$

  where H is the Heaviside function.

Cellular automata model

- The avalanches are continued until no site exceeds the threshold (which obviously happens after finitely many steps).
- As an example: (animated gif)



Continuum limit

- Passing to a continuum limit in

$$h_{ij}^{t+1} - h_{ij}^{t} = -M_{ij}^{kl} H(h_{ij}^{t} - h_{ij}^{c}), \quad \forall (i,j) \in \mathscr{O},$$

  gives (informally)

$$\frac{\partial}{\partial t} X(t,\xi) = \Delta H(X(t,\xi) - X^{c}(\xi)),$$

  where X is the continuous height-density function.
- In addition we impose zero Dirichlet boundary conditions:

$$H(X(t,\xi) - X^{c}(\xi)) = 0, \quad \text{on } \partial\mathscr{O}.$$

Continuum limit

- We will restrict to the supercritical case, i.e. supposing $x_0 \geq X^c$.
- Substituting $X \to X - X^c$ then yields

$$\frac{\partial}{\partial t} X(t,\xi) = \Delta H(X(t,\xi)),$$
$$X(0,\xi) = x_0(\xi)$$

with $x_0 \geq 0$ and zero Dirichlet boundary conditions:

$$H(X(t,\xi)) = 0, \quad \text{on } \partial\mathscr{O}.$$

- In other words $X^c \equiv 0$ without loss of generality.
- In this case also $X \geq 0$, so that equivalently

$$\frac{\partial}{\partial t} X(t,\xi) = \Delta \mathrm{sgn}(X(t,\xi)),$$
$$X(0,\xi) = x_0(\xi).$$

Continuum limit

- Types of sandpile models:

model	**evolution/threshold depending on**
critical height	height h
critical slope	first derivatives of height
Laplacian models	second derivatives of height

- Here we are considering a critical height model

Finite time extinction and SOC

Finite time extinction and self-organized criticality

Finite time extinction and SOC

- Self-organized criticality: We quote from [Bantay, Ianosi; Physica A, 1992]

> *"Criticality" refers to the power-law behavior of the spatial and temporal distributions, characteristic of critical phenomena.*
> *"Self-organized" refers to the fact that these systems naturally evolve into a critical state without any tuning of the external parameters, i.e. the critical state is an attractor of the dynamics.*

- Does SOC appear in

$$\frac{\partial}{\partial t}X(t,\xi) = \Delta \mathrm{sgn}(X(t,\xi))?$$

In other words, is the critical state $X^c \equiv 0$ attained in finite time.

Finite time extinction and SOC

Known results on finite time extinction:

- Consider

$$\frac{\partial}{\partial t} X(t,\xi) = \Delta\phi(X(t,\xi)), \quad t \geq 0,\ \xi \in \mathcal{O} \qquad (*)$$
$$\phi(X(t,\xi)) = 0, \quad \xi \in \partial\mathcal{O}.$$

- In [Diaz, Diaz; CPDE, 1979] finite time extinction (FTE) was first proven for (*). (More precisely, FTE for (*) was only formulated as a theorem for continuous ϕ, but it was observed that the techniques easily extend to multivalued ϕ including the sign function.)
- The main idea was to pass to a "dual" formulation by setting $Z = \Delta^{-1} X$:

$$\frac{\partial}{\partial t} Z(t,\xi) = \phi(\Delta Z(t,\xi)).$$

Then it is possible to construct explicit supersolutions to this dual formulation.

Finite time extinction and SOC

- In [Barbu; MMAS, 2012] another (more robust) approach based on energy methods was introduced.
- Informally the proof boils down to a combination of an L^1 and an L^∞ estimate of the solution:
- Informal L^∞ estimate:

$$\|X(t)\|_\infty \le \|x_0\|_\infty, \quad \forall t \ge 0.$$

- Informal L^1-estimate:

$$\begin{aligned}
\partial_t \int_{\mathcal{O}} |X(t,\xi)| d\xi &= \int_{\mathcal{O}} \mathrm{sgn}(X(t,\xi)) \Delta \mathrm{sgn}(X(t,\xi)) d\xi \\
&= -\int_{\mathcal{O}} |\nabla \mathrm{sgn}(X(t,\xi))|^2 d\xi \\
&\le -\left(\int_{\mathcal{O}} |\mathrm{sgn}(X(t,\xi))|^p d\xi \right)^{\frac{2}{p}} \\
&\le -(|\{\xi | X(t,\xi) \neq 0\}|)^{\frac{2}{p}},
\end{aligned}$$

for some (dimension dependent) $p > 2$. Note: $\frac{2}{p} < 1$.

Finite time extinction and SOC

- Observe

$$\int_{\mathcal{O}} |X(t,\xi)|d\xi \leq \|X(t)\|_\infty |\{\xi | X(t,\xi) \neq 0\}|.$$

- Using this above gives

$$\begin{aligned}\partial_t \int_{\mathcal{O}} |X(t,\xi)|d\xi &\leq -\frac{1}{\|X(t)\|_\infty^{\frac{2}{p}}} \left(\int_{\mathcal{O}} |X(t,\xi)|d\xi\right)^{\frac{2}{p}} \\ &\leq -\frac{1}{\|X_0\|_\infty^{\frac{2}{p}}} \left(\int_{\mathcal{O}} |X(t,\xi)|d\xi\right)^{\frac{2}{p}}.\end{aligned}$$

- It is easy to see that the solution of a singular ODE of the form

$$\dot{f} = -cf^\alpha, \quad \alpha \in (0,1),\ c > 0$$

vanishes in finite time.

The stochastic BTW model and some known results

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The stochastic BTW model

- In [Díaz-Guilera; EPL (Europhysics Letters), 1994], [Giacometti, Diaz-Guilera; Phys. Rev. E, 1998], [Díaz-Guilera; Phys. Rev. A, 1992] it was pointed out that it is more realistic to include stochastic perturbations.
- This leads to SPDE of the form

$$dX_t \in \Delta \mathrm{sgn}(X_t)dt + B(X_t)dW_t,$$

  with appropriate diffusion coefficients B.
- A related model to the (stochastic) BTW model is the (stochastic) Zhang model:

$$dX_t \in \Delta \mathrm{sgn}(X_t)dt + \delta\Delta X_t dt + B(X_t)dW_t.$$

- As compared to the BTW model, the Zhang model has additional diffusion $\delta\Delta X_t$. Since we work with Dirichlet boundary conditions, this helps in proving finite time extinction. In addition it makes the solutions more regular (since the dissipativity coefficient is non-degenerate).

The stochastic BTW model

- Finite time extinction can be reformulated in terms of the extinction time

$$\tau_0(\omega) := \inf\{t \geq 0 | X_t(\omega) = 0, \text{ a.e. in } \mathcal{O}\}.$$

We distinguish the following concepts:

(F1) Extinction with positive probability for small initial conditions: $\mathbb{P}[\tau_0 < \infty] > 0$, for small $X_0 = x_0$.

(F2) Extinction with positive probability: $\mathbb{P}[\tau_0 < \infty] > 0$, for all $X_0 = x_0$.

(F3) Finite time extinction: $\mathbb{P}[\tau_0 < \infty] = 1$, for all $X_0 = x_0$.

Some known results

- Existence and uniqueness of solutions to

$$dX_t \in \Delta\phi(X_t)dt + \sum_{k=1}^{N} f_k X_t d\beta_t^k$$

  with ϕ being possibly multi-valued goes back to [Barbu, Da Prato, Röckner; CMP, 2009].
- In the same paper (F1) for the Zhang model is shown for $d = 1$.
- In [Barbu, Da Prato, Röckner; JMAA, 2012] this was extended to prove (F1) for the BTW model for $d = 1$.
- In the recent work [Röckner, Wang; BiBoS-Preprint, 2011] finite time extinction for the Zhang model has been solved.
- In case of additive noise

$$dX_t \in \Delta \mathrm{sgn}(X_t)dt + dW_t,$$

  ergodicity has been shown for $d = 1$ in [Gess, Tölle; JMPA, to appear].
- In [Barbu, Röckner; ARMA, 2013] (F1) has been shown for the related stochastic total variation flow for $d \leq 3$.

Finite time extinction for the stochastic BTW model

Finite time extinction for the stochastic BTW model

The stochastic BTW model

Recall: We aim to prove finite time extinction for

$$dX_t \in \Delta \mathrm{sgn}(X_t)dt + \sum_{k=1}^{N} f_k X_t d\beta_t^k \qquad \text{(BTW)}$$

$$X_0 = x_0,$$

with zero Dirichlet boundary conditions.

Transformation

- Our approach to FTE will be based on considering the following transformation: Set $\mu_t := \sum_{k=1}^N f_k \beta_t^k$, $\tilde{\mu} := \sum_{k=1}^N f_k^2$ and $Y_t := e^{-\mu_t} X_t$. An informal calculation shows
$$\partial Y_t \in e^{\mu_t} \Delta \mathrm{sgn}(Y_t) - \tilde{\mu} Y_t. \qquad (*)$$

- We show rigorously: Given a solution Y to $(*)$, $X_t := e^{\mu_t} Y_t$ is the unique solution to
$$dX_t \in \Delta \mathrm{sgn}(X_t) dt + \sum_{k=1}^N f_k X_t d\beta_t^k.$$

Outline of the proof

- There are two main ingredients of the proof:
  1. A uniform control on $\|X_t\|_p$ for all $p \geq 1$.
  2. An energy inequality for $\|Y_t\|_1$ (roughly speaking).
- On an intuitive level the arguments become clear by approximating

$$r^{[m]} := |r|^{m-1} r \to \mathrm{sgn}, \quad \text{for } m \downarrow 0.$$

  To make these arguments rigorous, in fact a different (non-singular, non-degenerate) approximation of sgn is used.
- In the following let Y_t be a solution to

$$\partial_t Y_t \in e^{\mu_t} \Delta Y_t^{[m]} - \tilde{\mu} Y_t.$$

Step 1: Informal L^p bound

- **Step 1:** A uniform control on $\|X_t\|_p$ for all $p \geq 1$.
- We may informally compute for all $p \geq 1$:

$$\begin{aligned}\partial_t \int_{\mathscr{O}} e^{p\tilde{\mu}t}|Y_t|^p d\xi =& p\int_{\mathscr{O}} e^{p\tilde{\mu}t} Y_t^{[p-1]} e^{\mu_t}\Delta Y_t^{[m]} d\xi \\ =& -\frac{4(p-1)mp}{(p+m-1)^2}\int_{\mathscr{O}} e^{\mu_t+p\tilde{\mu}t}\left(\nabla |Y_t|^{\frac{p+m-1}{2}}\right)^2 d\xi \\ &+\frac{pm}{p+m-1}\int_{\mathscr{O}} |Y_t|^{p+m-1}\Delta e^{\mu_t+p\tilde{\mu}t} d\xi .\end{aligned}$$

- Taking $p > 1$ and then $m \to 0$ we may "deduce" from this

$$\partial_t \int_{\mathscr{O}} e^{p\tilde{\mu}t}|Y_t|^p d\xi \leq 0.$$

Step 1: Informal L^p bound

- In order to turn the above bound on Y into a bound on X we note

$$\begin{aligned}\int_{\mathscr{O}} |X_t|^p d\xi &= \int_{\mathscr{O}} e^{-p\mu_t} |Y_t|^p d\xi \\ &= \int_{\mathscr{O}} e^{-p\mu_t - p\tilde{\mu} t} e^{p\tilde{\mu} t} |Y_t|^p d\xi.\end{aligned}$$

- Under the assumption $\tilde{\mu} = \sum_{k=1}^N f_k^2 > 0$ we have

$$\|e^{-\mu_t - \tilde{\mu} t}\|_{L^\infty(\mathscr{O})} = \|e^{\sum_{k=1}^N f_k(\xi)\beta_t^k - f_k(\xi)^2 t}\|_{L^\infty(\mathscr{O})} \le C(\omega) < \infty.$$

- Hence,

$$\begin{aligned}\int_{\mathscr{O}} |X_t|^p d\xi &\le C^p(\omega) \int_{\mathscr{O}} e^{p\tilde{\mu} t} |Y_t|^p d\xi \\ &\le C^p(\omega) \int_{\mathscr{O}} |x_0|^p d\xi.\end{aligned}$$

- We conclude

$$\|X_t\|_p \le C(\omega) \|x_0\|_p.$$

Step 2: Informal "L^1" bound

- **Step 2:** An energy inequality for $\|Y_t\|_1$ (roughly speaking).
- Recall:

$$\partial_t \int_{\mathscr{O}} e^{p\tilde{\mu}t}|Y_t|^p d\xi = -\frac{4(p-1)mp}{(p+m-1)^2}\int_{\mathscr{O}} e^{\mu_t+p\tilde{\mu}t}\left(\nabla|Y_t|^{\frac{p+m-1}{2}}\right)^2 d\xi + \frac{pm}{p+m-1}\int_{\mathscr{O}}|Y_t|^{p+m-1}\Delta e^{\mu_t+p\tilde{\mu}t}d\xi,\ p\geq 1.$$

- Choose $p=m+1$ and let $m\to 0$. We obtain

$$\partial_t \int_{\mathscr{O}} e^{\tilde{\mu}t}|Y_t|d\xi = -\int_{\mathscr{O}} e^{\mu_t+\tilde{\mu}t}(\nabla\eta)^2 d\xi + \frac{1}{2}\int_{\mathscr{O}}\eta^2\Delta e^{\mu_t+\tilde{\mu}t}d\xi,$$

where η is a selection from $\text{sgn}(Y)$, i.e. $\eta_t(\xi)\in\text{sgn}(Y_t(\xi))$ a.e..

Step 2: Informal "L^1" bound

- Let φ be the classical solution to

$$\begin{aligned} \Delta\varphi &= -1, \quad \text{on } \mathcal{O} \\ \varphi &= 1, \quad \text{on } \partial\mathcal{O}. \end{aligned}$$

  Note $1 \le \varphi \le \|\varphi\|_\infty =: C_\varphi$.
- We informally compute

$$\partial_t \int_{\mathcal{O}} \varphi e^{\tilde{\mu} t} |Y_t| d\xi = -\int_{\mathcal{O}} \varphi e^{\mu_t + \tilde{\mu} t} (\nabla \eta)^2 d\xi + \frac{1}{2} \int_{\mathcal{O}} \eta^2 \Delta(\varphi e^{\mu_t + \tilde{\mu} t}) d\xi.$$

  Note that the perturbation term has negative sign for small times!

Step 2: Informal "L^1" bound

- We informally compute

$$\partial_t \int_{\mathcal{O}} e^{-\mu_s} \varphi |Y_t|^p d\xi = - \frac{4(p-1)mp}{(p+m-1)^2} \int_{\mathcal{O}} e^{\mu_t - \mu_s} \varphi \left(\nabla |Y_t|^{\frac{p+m-1}{2}} \right)^2 d\xi + \frac{pm}{p+m-1} \int_{\mathcal{O}} |Y_t|^{p+m-1} \Delta e^{\mu_t - \mu_s} \varphi d\xi .$$

- Taking $p = m+1$, $m \to 0$ we get

$$\partial_t \int_{\mathcal{O}} e^{-\mu_s} \varphi |Y_t| d\xi = - \int_{\mathcal{O}} e^{\mu_t - \mu_s} \varphi (\nabla \eta)^2 d\xi + \frac{1}{2} \int_{\mathcal{O}} \eta^2 \Delta e^{\mu_t - \mu_s} \varphi d\xi .$$

Step 3: Finishing the proof

- **Step 3:** Deducing finite time extinction
- For the sake of simplicity let us restrict to $d = 1$. In this case $H_0^1 \hookrightarrow L^\infty$.
- Recall: On intervals $[s,t]$ on which $\Delta e^{\mu - \mu_s}\varphi \le 0$ we have

$$\partial_t \int_{\mathcal{O}} e^{-\mu_s}\varphi |Y_t| d\xi \le -\int_{\mathcal{O}} e^{\mu_t - \mu_s}\varphi (\nabla \eta)^2 \, d\xi.$$

- Hence,

$$\int_{\mathcal{O}} e^{-\mu_s}\varphi |Y_t| d\xi \le \int_{\mathcal{O}} e^{-\mu_s}\varphi |Y_s| d\xi - \int_s^t \left(\inf_{\xi \in \mathcal{O}} e^{\mu_r - \mu_s} \right) \|\eta_r\|_\infty dr.$$

By step one we observe

$$\int_{\mathcal{O}} e^{-\mu_s}\varphi |Y_s| \le C(\omega) C_\varphi \|x_0\|_1.$$

Step 3: Finishing the proof

- Moreover, since $\|\eta_t\|_\infty = 0$ implies $Y_t \equiv 0$ we may deduce

$$\int_{\mathscr{O}} e^{-\mu_s} \varphi |Y_t| d\xi \le C(\omega) C_\varphi \|x_0\|_1 - \int_s^t \left(\inf_{\xi \in \mathscr{O}} e^{\mu_r - \mu_s} \right) dr \vee 0, \qquad (*)$$

  for all intervals $[s,t]$ such that $\Delta e^{\mu - \mu_s} \varphi$ is non-positive.
- Since

$$\|\mu_t - \mu_s\|_{C^2(\mathscr{O})} \le \left(\sum_{k=1}^N \|f_k\|_{C^2(\mathscr{O})} \right) |\beta_t - \beta_s|$$

  for $\Delta e^{\mu - \mu_s} \varphi$ to be non-positive we have to restrict to intervals $[s,t]$ where $|\beta_t - \beta_s|$ remains small.
- Due to properties of Brownian motion we may find such intervals $[s,t]$ of arbitrary length.
- Hence (*) implies finite time extinction (with extinction time t_0 depending on x_0 only via its L^1-norm).

Main result

Theorem

Let $x_0 \in L^\infty(\mathscr{O})$, X *be the unique variational solution to BTW and let*

$$\tau_0(\omega) := \inf\{t \geq 0 | X_t(\omega) = 0, \text{ for a.e. } \xi \in \mathscr{O}\}.$$

Then finite time extinction holds, i.e.

$$\mathbb{P}[\tau_0 < \infty] = 1.$$

If $d = 1$, *the extinction time* $\tau_0(\omega)$ *may be chosen uniformly for* x_0 *bounded in* $L^1(\mathscr{O})$. *For* $d \geq 2$ *and every* $p > \frac{d}{2}$, *the extinction time* $\tau_0(\omega)$ *may be chosen uniformly for* x_0 *bounded in* $L^p(\mathscr{O})$.

Exponential decay

Exponential decay

Exponential decay

- Using similar ideas as in the proof of the main Theorem we (partially) sharpen a result obtained in [Barbu, Röckner; CMP, 2012].
- Assuming $\tilde{\mu}=\sum_{k=1}^{N} f_k^2>0$ it had been shown:

$$\int_K X_t d\xi \le \|x_0\|_2 |K|^{\frac{1}{2}} e^{\sup_K \tilde{\mu}^{\frac{1}{2}}} \left(\sum_{k=1}^{N}\beta_k(t)\right)^{\frac{1}{2}} e^{-\frac{t}{2}\inf_{K'}\tilde{\mu}},$$

for every compact set $K\subseteq\mathscr{O}$ and every compact neighborhood $K'\supseteq K$.

Theorem

Let $x_0\in L^\infty(\mathscr{O})$ and let X be the corresponding unique solution to (BTW). Then,

$$X_t \le e^{\mu_t-\tilde{\mu}t}\|x_0\|_\infty, \quad \forall t\in\mathbb{R}_+, \text{ a.e. } \xi\in\mathscr{O}.$$

- idea of the proof: Let $K=\|x_0\|_\infty e^{-t\tilde{\mu}}$. Since $K>0$ we have $\operatorname{sgn}(K)\equiv 1$ and thus

$$e^{\mu}\Delta\operatorname{sgn}(K)-\tilde{\mu}K=-\tilde{\mu}K=\partial_t K.$$

I.e. K is a supersolution and we conclude $Y\le K$.

Thanks

Thanks!