

Finite time extinction for stochastic sign fast diffusion and self-organized criticality.

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Outline

- 1 Self-organized criticality
- 2 Derivation of the BTW model from a cellular automaton
- 3 Finite time extinction and self-organized criticality
- 4 Finite time extinction for stochastic BTW

Self-organized criticality

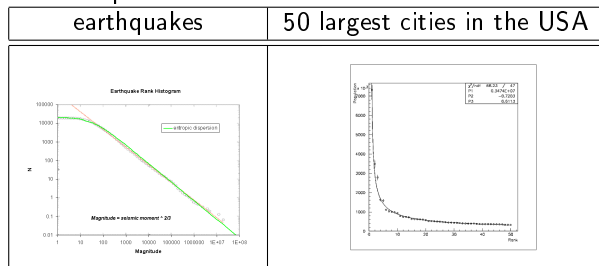
Self-organized criticality

Self-organized criticality

- Many (complex) systems in nature exhibit power law scaling: The number of an event $N(s)$ scales with the event size s as

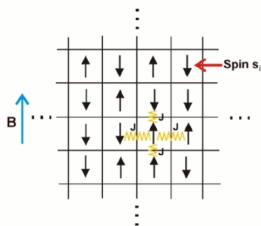
$$N(s) \sim s^{-\alpha}$$

- For example:



Self-organized criticality

- Phase-transitions: The Ising model, ferromagnetism



- Correlation length ξ :

$$\xi \sim \left(\frac{T - T_c}{T} \right)^{-\nu}$$

- Critical temperature $T = T_c$:
 - strongly correlated: small perturbations can have global effects
 - no specific length scale (complex system, criticality)
- Observe: For $T = T_c$, power-law scaling for $N(s)$ being the number of $+1$ clusters of size s .

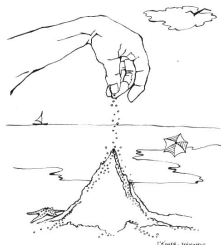
Self-organized criticality

- Ising model needs precise tuning $T = T_c$ to display power law scaling
- How can this occur in nature?
- Idea of self-organized criticality: [Bantay, Ianozi; Physica A, 1992]

“Criticality” refers to the power-law behavior of the spatial and temporal distributions, characteristic of critical phenomena.

“Self-organized” refers to the fact that these systems naturally evolve into a critical state without any tuning of the external parameters, i.e. the critical state is an attractor of the dynamics.

- Bak, Tang, Wiesenfeld: Sandpile as a toy model of self-organized criticality



Sandpiles

- Two scales: Slow energy injection (adding sand), fast energy diffusion (avalanches)
- Criticality: No typical avalanche size, local perturbation may have global effects
- Power law scaling: $N(s)$ is the number of avalanches of size s

TYPICAL AVALANCHES

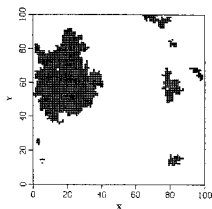


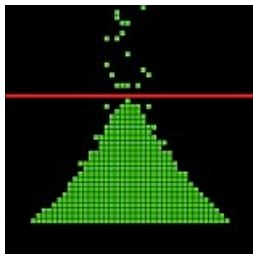
FIGURE 2

Derivation of the BTW model from a cellular automaton

Derivation of the BTW model from a cellular automaton

Cellular automata model

- The following model goes back to [Bantay, Iannosi; Physica A, 1992].
- Cellular automata model for a sandpile:



Cellular automata model

- Consider an $N \times N$ square lattice, representing a discrete region $\mathcal{O} = \{(i,j)\}_{i,j=1}^N$.
- At each site (i,j) the height of the sandpile at time t is h_{ij}^t .
- The system is perturbed externally until the height h exceeds a threshold (critical) value h^c .
- Then, a toppling (avalanche) event occurs: The toppling at any 'activated' site (k,l) is described by:

$$h_{ij}^{t+1} \rightarrow h_{ij}^t - M_{ij}^{kl}, \quad \forall (i,j) \in \mathcal{O},$$

where

$$M_{ij}^{kl} = \begin{cases} 4 & (k,l) = (i,j) \\ -1 & (k,l) \sim (i,j) \\ 0 & \text{otherwise.} \end{cases}$$

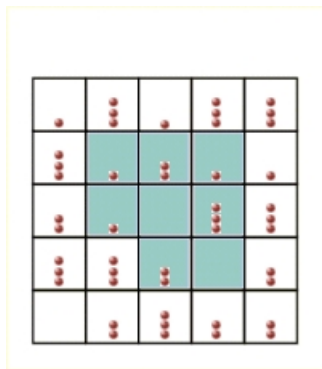
- Rewrite as:

$$h_{ij}^{t+1} - h_{ij}^t = -M_{ij}^{kl} H(h_{ij}^t - h_{ij}^c), \quad \forall (i,j) \in \mathcal{O},$$

where H is the Heaviside function.

Cellular automata model

- The avalanches are continued until no site exceeds the threshold (which obviously happens after finitely many steps).
- As an example: (animated gif)



Continuum limit

- Passing to a continuum limit in

$$h_{ij}^{t+1} - h_{ij}^t = -M_{ij}^{kl} H(h_{ij}^t - h_{ij}^c), \quad \forall (i,j) \in \mathcal{O},$$

gives (informally)

$$\frac{\partial}{\partial t} X(t, \xi) = \Delta H(X(t, \xi) - X^c(\xi)),$$

where X is the continuous height-density function.

- In addition we impose zero Dirichlet boundary conditions:

$$H(X(t, \xi) - X^c(\xi)) = 0, \quad \text{on } \partial\mathcal{O}.$$

- Note: Only the relaxation/diffusion part modeled here. For full SOC-model we would have to include the external, random energy input.

The stochastic BTW model

- In [Díaz-Guilera; EPL (Europhysics Letters), 1994], [Giacometti, Diaz-Guilera; Phys. Rev. E, 1998], [Díaz-Guilera; Phys. Rev. A, 1992] it was pointed out that it is more realistic to include stochastic perturbations.
- This leads to SPDE of the form

$$dX_t = \Delta H(X_t - X^c) + B(X_t - X^c)dW_t,$$

with appropriate diffusion coefficients B .

- We study linear multiplicative noise, i.e.

$$dX_t = \Delta H(X_t - X^c) + \sum_{k=1}^N f_k(X_t - X^c)d\beta_t^k.$$

- Question: Do avalanches end in finite time?

Continuum limit

- Types of sandpile models:

model	evolution/threshold depending on
critical height	height h
critical slope	first derivatives of height
Laplacian models	second derivatives of height

- Here we are considering a critical height model

Finite time extinction and SOC

Finite time extinction and self-organized criticality

Finite time extinction and SOC

- We concentrate on the relaxation/diffusion of energy part. Question: Do avalanches end in finite time?
- Recall:

$$\frac{\partial}{\partial t} X(t, \xi) = \Delta H(X(t, \xi) - X^c(\xi)),$$

- We will restrict to the supercritical case, i.e. supposing $x_0 \geq X^c$.
- Substituting $X \rightarrow X - X^c$ then yields

$$\begin{aligned} \frac{\partial}{\partial t} X(t, \xi) &= \Delta H(X(t, \xi)), \\ X(0, \xi) &= x_0(\xi) \end{aligned}$$

with $x_0 \geq 0$ and zero Dirichlet boundary conditions:

$$H(X(t, \xi)) = 0, \quad \text{on } \partial\mathcal{O}.$$

- In other words $X^c \equiv 0$ without loss of generality.

Finite time extinction and SOC

- In this case also $X \geq 0$, so that equivalently

$$\begin{aligned}\frac{\partial}{\partial t} X(t, \xi) &= \Delta \operatorname{sgn}(X(t, \xi)), \\ X(0, \xi) &= x_0(\xi).\end{aligned}$$

- Does SOC appear in

$$\frac{\partial}{\partial t} X(t, \xi) = \Delta \operatorname{sgn}(X(t, \xi))?$$

In other words, is the critical state $X^c \equiv 0$ attained in finite time.

Finite time extinction for deterministic PDE

Finite time extinction for deterministic PDE

Finite time extinction for singular ODE

- Consider the singular ODE

$$\dot{f} = -cf^\alpha, \quad \alpha \in (0,1), \quad c > 0.$$

- Then:

$$(f^{1-\alpha})' = -(1-\alpha).$$

- We obtain

$$f^{1-\alpha}(t) = f^{1-\alpha}(0) - (1-\alpha)ct$$

which implies finite time extinction.

Finite time extinction and SOC

- [Diaz, Diaz; CPDE, 1979] finite time extinction (FTE) was first proven for

$$\frac{\partial}{\partial t} X(t, \xi) = \Delta \operatorname{sgn}(X(t, \xi)).$$

- In [Barbu; MMAS, 2012] another (more robust) approach based on energy methods was introduced.
- Informally the proof boils down to a combination of an L^1 and an L^∞ estimate of the solution:
- Informal L^∞ estimate:

$$\|X(t)\|_\infty \leq \|x_0\|_\infty, \quad \forall t \geq 0.$$

- Informal L^1 -estimate:

$$\begin{aligned} \partial_t \int_{\mathcal{O}} |X(t, \xi)| d\xi &= \int_{\mathcal{O}} \operatorname{sgn}(X(t, \xi)) \Delta \operatorname{sgn}(X(t, \xi)) d\xi \\ &= - \int_{\mathcal{O}} |\nabla \operatorname{sgn}(X(t, \xi))|^2 d\xi \\ &\leq - \left(\int_{\mathcal{O}} |\operatorname{sgn}(X(t, \xi))|^p d\xi \right)^{\frac{2}{p}} \end{aligned}$$

Finite time extinction and SOC

- Observe

$$\int_{\mathcal{O}} |X(t, \xi)| d\xi \leq \|X(t)\|_{\infty} |\{\xi | X(t, \xi) \neq 0\}|.$$

- Using this above gives

$$\begin{aligned} \partial_t \int_{\mathcal{O}} |X(t, \xi)| d\xi &\leq -\frac{1}{\|X(t)\|_{\infty}^{\frac{2}{p}}} \left(\int_{\mathcal{O}} |X(t, \xi)| d\xi \right)^{\frac{2}{p}} \\ &\leq -\frac{1}{\|X_0\|_{\infty}^{\frac{2}{p}}} \left(\int_{\mathcal{O}} |X(t, \xi)| d\xi \right)^{\frac{2}{p}}. \end{aligned}$$

- We are left with the singular ODE

$$\dot{f} = -cf^{\alpha}, \quad \alpha \in (0, 1), \quad c > 0$$

for which we have seen that finite time extinction holds.

Finite time extinction for stochastic BTW

Finite time extinction for stochastic BTW

The stochastic BTW model

- Recall:

$$dX_t = \Delta \operatorname{sgn}(X_t) + \sum_{k=1}^N f_k X_t d\beta_t^k,$$

with zero Dirichlet boundary conditions.

- Finite time extinction can be reformulated in terms of the extinction time

$$\tau_0(\omega) := \inf\{t \geq 0 \mid X_t(\omega) = 0, \text{ a.e. in } \mathcal{O}\}.$$

We distinguish the following concepts:

- (F1) Extinction with positive probability for small initial conditions: $\mathbb{P}[\tau_0 < \infty] > 0$, for small $X_0 = x_0$.
- (F2) Extinction with positive probability: $\mathbb{P}[\tau_0 < \infty] > 0$, for all $X_0 = x_0$.
- (F3) Finite time extinction: $\mathbb{P}[\tau_0 < \infty] = 1$, for all $X_0 = x_0$.

Some known results

- Existence and uniqueness of solutions to

$$dX_t \in \Delta\phi(X_t)dt + \sum_{k=1}^N f_k X_t d\beta_t^k$$

with ϕ being possibly multi-valued goes back to [Barbu, Da Prato, Röckner; CMP, 2009].

- In the same paper (F1) for the Zhang model is shown for $d = 1$.
- In [Barbu, Da Prato, Röckner; JMAA, 2012] this was extended to prove (F1) for the BTW model for $d = 1$.
- In the recent work [Röckner, Wang; BiBoS-Preprint, 2011] finite time extinction for the Zhang model has been solved.
- In case of additive noise

$$dX_t \in \Delta\text{sgn}(X_t)dt + dW_t,$$

ergodicity has been shown for $d = 1$ in [Gess, Tölle; JMPA, to appear].

- In [Barbu, Röckner; ARMA, 2013] (F1) has been shown for the related stochastic total variation flow for $d \leq 3$.

Transformation

- Recall:

$$dX_t = \Delta \operatorname{sgn}(X_t) + \sum_{k=1}^N f_k X_t d\beta_t^k,$$

- Our approach to FTE will be based on considering the following transformation: Set $\mu_t := \sum_{k=1}^N f_k \beta_t^k$, $\tilde{\mu} := \sum_{k=1}^N f_k^2$ and $Y_t := e^{-\mu_t} X_t$. An informal calculation shows

$$\partial Y_t \in e^{\mu_t} \Delta \operatorname{sgn}(Y_t) - \tilde{\mu} Y_t. \quad (*)$$

- Compare the deterministic setting:

$$\partial Y_t \in \Delta \operatorname{sgn}(Y_t).$$

Outline of the proof

- There are two main ingredients of the proof:
 - ① A uniform control on $\|X_t\|_p$ for all $p \geq 1$.
 - ② An energy inequality for a weighted L^1 -norm.
- On an intuitive level the arguments become clear by approximating

$$r^{[m]} := |r|^{m-1}r \rightarrow \text{sgn}, \quad \text{for } m \downarrow 0.$$

To make these arguments rigorous, in fact a different (non-singular, non-degenerate) approximation of sgn is used.

- In the following let Y_t be a solution to

$$\partial_t Y_t \in e^{\mu t} \Delta Y_t^{[m]} - \tilde{\mu} Y_t.$$

Step 1: Informal L^p bound

- **Step 1:** A uniform control on $\|X_t\|_p$ for all $p \geq 1$.
- We may informally compute for all $p \geq 1$:

$$\begin{aligned} \partial_t \int_{\mathcal{O}} |Y_t|^p d\xi &= p \int_{\mathcal{O}} Y_t^{[p-1]} e^{\mu t} \Delta Y_t^{[m]} d\xi \\ &= -\frac{4(p-1)mp}{(p+m-1)^2} \int_{\mathcal{O}} e^{\mu t} \left(\nabla |Y_t|^{\frac{p+m-1}{2}} \right)^2 d\xi \\ &\quad + \frac{pm}{p+m-1} \int_{\mathcal{O}} |Y_t|^{p+m-1} \Delta e^{\mu t} d\xi. \end{aligned}$$

- Taking $p > 1$ and then $m \rightarrow 0$ we may “deduce” from this

$$\partial_t \int_{\mathcal{O}} |Y_t|^p d\xi \leq 0.$$

Step 2: Informal “ L^1 ” bound

- **Step 2:** An energy inequality for a weighted L^1 -norm.

$$\begin{aligned} \partial_t \int_{\mathcal{O}} |Y_t|^p d\xi &= -\frac{4(p-1)mp}{(p+m-1)^2} \int_{\mathcal{O}} e^{\mu t} \left(\nabla |Y_t|^{\frac{p+m-1}{2}} \right)^2 d\xi \\ &\quad + \frac{pm}{p+m-1} \int_{\mathcal{O}} |Y_t|^{p+m-1} \Delta e^{\mu t} d\xi, \quad p \geq 1. \end{aligned}$$

- Choose $p = m + 1$ and let $m \rightarrow 0$. We obtain

$$\partial_t \int_{\mathcal{O}} |Y_t| d\xi = - \int_{\mathcal{O}} e^{\mu t} (\nabla \operatorname{sgn}(Y_t))^2 d\xi + \frac{1}{2} \int_{\mathcal{O}} \Delta e^{\mu t} d\xi$$

- Recall: deterministic case

$$\partial_t \int_{\mathcal{O}} |Y_t| d\xi = - \int_{\mathcal{O}} |\nabla \operatorname{sgn}(Y_t)|^2 d\xi.$$

Step 2: Informal “ L^1 ” bound

Key trick: Use a weighted L^1 -norm

- Let φ be the classical solution to

$$\begin{aligned}\Delta\varphi &= -1, & \text{on } \mathcal{O} \\ \varphi &= 1, & \text{on } \partial\mathcal{O}.\end{aligned}$$

Note $1 \leq \varphi \leq \|\varphi\|_\infty =: C_\varphi$.

- We informally compute

$$\partial_t \int_{\mathcal{O}} \varphi |Y_t| d\xi = - \int_{\mathcal{O}} \varphi e^{\mu t} (\nabla \text{sgn}(Y_t))^2 d\xi + \frac{1}{2} \int_{\mathcal{O}} \Delta(\varphi e^{\mu t}) d\xi.$$

Note that the perturbation term has negative sign for small times!

- Shift the initial time

$$\partial_t \int_{\mathcal{O}} e^{-\mu s} \varphi |Y_t| d\xi = - \int_{\mathcal{O}} e^{\mu t - \mu s} \varphi (\nabla \text{sgn}(Y_t))^2 d\xi + \frac{1}{2} \int_{\mathcal{O}} \text{sgn}(Y_t)^2 \Delta e^{\mu t - \mu s} \varphi d\xi$$

Main result

Theorem

Let $x_0 \in L^\infty(\mathcal{O})$, X be the unique variational solution to BTW and let

$$\tau_0(\omega) := \inf\{t \geq 0 \mid X_t(\omega) = 0, \text{ for a.e. } \xi \in \mathcal{O}\}.$$

Then finite time extinction holds, i.e.

$$\mathbb{P}[\tau_0 < \infty] = 1.$$

For every $p > \frac{d}{2} \vee 1$, the extinction time $\tau_0(\omega)$ may be chosen uniformly for x_0 bounded in $L^p(\mathcal{O})$.

Thanks

Thanks!