Finite speed of propagation for stochastic porous media equations.

Benjamin Gess[1]

[1]Institut für Mathematik,
Humboldt-Universität zu Berlin

Stochastics and Real World Models - IGK workshop,
Bielefeld, July 2013

preprint:[arXiv:1210.2415],
to appear in SIAM Journal on Mathematical Analysis (SIMA).

Outline

The deterministic case

The deterministic case

The Deterministic PME

- Recall: The deterministic porous medium equation

$$\frac{d}{dt}u = \Delta\left(u^m\right), \quad m > 1 \tag{1}$$

for non-negative initial conditions $u_0 \geq 0$ [Vázquez, 2006].
For simplicity we write u^m for $|u|^{m-1}u$, even if u is not necessarily non-negative.

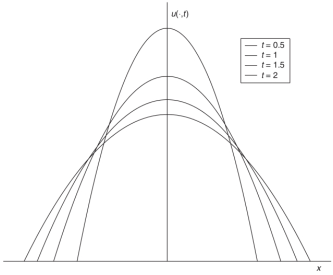
- In the superlinear case $m > 1$, (1) has degenerate diffusivity:

$$\frac{d}{dt}u = mu^{m-1}\Delta u + m(m-1)u^{m-2}|\nabla u|^2.$$

The diffusivity coefficient vanishes for $u \to 0$.

Finite speed of propagation

- Limited regularity: ∇u discontinuous. E.g. Barenblatt solutions:



- Regularity is limited precisely at the free boundary.
- Comparison to Barenblatt solutions yields finite speed of propagation.

Optimal estimates

- Finite speed of propagation is a local property.
- *Finite speed of hole-filling*:
  Let u_0 vanish in $B_R(x_0)$. Then u vanishes in $B_{R(t)}(x_0)$ with

$$R(t) \geq R - \left(\frac{\|u\|_\infty^{m-1}}{C_{det}} t \right)^{1/2}.$$

  This is optimal.
- *Finite speed of propagation:*
  Let $S(t) := \{x \in \mathbb{R}^d | u(t,x) > 0\}$ be the positivity set. Then

$$S(t+h) \subseteq B_{ch^{1/2}}(S(t)).$$

The stochastic case

The stochastic case

Stochastic porous medium equation

- Stochastic porous medium equation

$$dX_t = \Delta X_t^m dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^{(k)}. \qquad \text{(SPME)}$$

on bounded domains, homogeneous Dirichlet boundary conditions.

- β^k Brownian motion, $f_k \in C^\infty(\mathscr{O})$.

Known results

- *Finite speed of hole-filling* [Barbu, Röckner, EJP 2012]: Let X_0 vanish in $B_R(x_0)$. Then X vanishes in $B_{R(t,\omega)}(x_0)$ for some function $R(\cdot,\omega):[0,T]\to(0,R)$.
- No uniform control on $R(t,\omega)$ in x_0 $\to$ cannot deduce finite speed of propagation
- No information about optimality of the bounds
- Two aims:
  - show finite speed of propagation
  - deduce (locally) optimal bounds

Transformation

- Recall:

$$dX_t = \Delta\left(X_t^m\right) dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^{(k)}.$$

- Set $Y_t := e^{\mu_t} X_t$, where $\mu_t = -\sum_{k=1}^{N} f_k \beta_t^{(k)}$. Then

$$\partial_t Y(t,x) = e^{\mu(t,x)} \Delta\left(e^{-\mu(t,x)} Y(t,x)\right)^m.$$

- Existence and uniqueness in [Barbu, Röckner, JDE, 2011], [G., arXiv:1108.2413; to appear in AoP].

Non-spatially distributed noise

- Assume f_k constant. Then

$$\partial_t Y_t = e^{(1-m)\mu_t} \Delta Y_t^m.$$

- Let $F' := e^{(1-m)\mu_t}$, $g = F^{-1}$. Then $u_t := Y_{g(t)}$ solves

$$\frac{d}{dt} u = \Delta (u^m)$$

- Finite speed of propagation follows from the deterministic case and the estimates are **optimal**.
- *Finite speed of hole-filling:*
  Let X_0 vanish in $B_R(x_0)$. Then X_t vanishes in $B_{R_{stoch}(t,\omega)}(x_0)$ with

$$\begin{aligned} R_{stoch}(t) &= R - \left(\frac{H^{m-1}}{C_{det}} F(t) \right)^{\frac{1}{2}} \\ &= R - \left(\frac{H^{m-1}}{C_{det}} \int_0^t e^{-(m-1)\mu_r(\omega)} dr \right)^{\frac{1}{2}}. \end{aligned}$$

Spatially distributed noise

- Recall:

$$\partial_t Y(t,x) = e^{\mu(t,x)} \Delta \left(e^{-\mu(t,x)} Y(t,x) \right)^m.$$

- freeze coefficients in space:

$$\partial_t Y(t,x) \approx e^{\mu(t,x_0)} \Delta \left(e^{-\mu(t,x_0)} Y(t,x) \right)^m,$$

  on small balls $B_r(x_0)$.

- freeze coefficients in time:

$$\partial_t Y(t,x) \approx \Delta Y(t,x)^m,$$

  for small times $t \approx 0$.

Hole-filling

- *Finite speed of hole-filling:*
  Let X_0 vanish in $B_R(x_0)$. Then X_t vanishes in $B_{R_{stoch}(t,\omega)}(x_0)$ with

$$R_{stoch}(t,\omega) = R - \left(\frac{H^{m-1}}{C_{det}} F(t,\omega)\right)^{\frac{1}{2}} C_R^{-\frac{1}{2}}(\omega)$$
$$= R - \left(\frac{H^{m-1}}{C_{det}} \int_0^t e^{-(m-1)\mu_r(x_0,\omega)} dr\right)^{\frac{1}{2}} C_R^{-\frac{1}{2}}(\omega),$$

  with $\lim_{R\downarrow 0} C_R = 1$.
- For $R \approx 0$ we recover the optimal rate from the spatially homogeneous case with $\mu_r \equiv \mu_r(\xi_0)$.

Hole-filling

- *Finite speed of hole-filling:*
  Let X_0 vanish in $B_R(x_0)$. Then X_t vanishes in $B_{R_{stoch}(t,\omega)}(x_0)$ with

$$R_{stoch}(t,\omega) = R - \left(\frac{H^{m-1}}{C_{det}} t\right)^{\frac{1}{2}} \sqrt{C_t(\omega)}.$$

  with $\lim_{t\downarrow 0} C_t = 1$.
- For $t \approx 0$ we recover the optimal rate from the deterministic case.

Finite speed of propagation

- *Finite speed of propagation:*
  Let X be an essentially bounded, non-negative solution to (SPME). Then,

$$supp(X_t) \subseteq B_{\sqrt{t}\left(\frac{H^{m-1}}{C_{det}}\right)^{\frac{1}{2}}\sqrt{C_t(\omega)}}(\mathrm{supp}(X_0)), \quad \forall t \in [0,T],$$

with $C_t \to 1$ for $t \to 0$.

Stabilization by noise

Stabilization by noise

Deterministic case

Semilinear case:

- Chafee-Infante

$$dX_t = \Delta X_t dt + \lambda X_t dt - X_t^3 dt, \quad \lambda > 0.$$

- There is an associated attractor with finite fractal dimension (dep. on λ).

Porous medium:

- Linear perturbation

$$\frac{d}{dt} u = \Delta(u^m) + \lambda u, \tag{2}$$

with $\lambda \geq 0$, bounded domain, homogeneous Dirichlet boundary conditions.

- The attractor to (2) has infinite fractal dimension iff $\lambda > 0$ [Efendiev, Zelik, Proc. Lond. Math. Soc., 2008].

Additive noise

Stochastic case

Additive noise

Additive noise

Semilinear-case:

- Chafee-Infante

$$dX_t = \Delta X_t dt + \lambda X_t dt - X_t^3 dt + dW_t, \quad \lambda > 0.$$

- If W_t is “sufficiently non-degenerate” then the random attractor consists of a single point, i.e. stabilization by noise.

Porous medium:

- PME perturbed by additive noise

$$dX_t = (\Delta(X_t^m) + \lambda X_t)dt + dW_t, \quad \lambda \geq 0. \tag{3}$$

- If $d \leq 2$, noise “non-degenerate”, then (3) has a random attractor consisting of a single (random) point [G., JDDE, 2013].

Multiplicative noise

Stochastic case

Multiplicative noise

Multiplicative noise

Linear-case:

- There are linear operators B_k such that

$$dX_t = \Delta X_t dt + \lambda X_t dt + \sum_{k=1}^{\infty} B_k X_t \circ d\beta_t^k,$$

  becomes exponentially stable. In particular, $\mathscr{A}(\omega) = \{0\}$. [Caraballo, Robinson; Systems & Control Letters (2004)]

Porous medium:

- PME perturbed by linear multiplicative noise

$$dX_t = (\Delta(X_t^m) + \lambda X_t)\, dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^{(k)}. \tag{4}$$

- The random attractor for (4) has infinite fractal dimension iff $\lambda > 0$.

Idea of the proof

Semilinear case:

- Consider:

$$dX_t = \Delta X_t dt + \lambda X_t dt - X_t^3 dt, \quad \lambda > 0.$$

Lower bounds on the dimension of the attractor are proven via linearization:

$$dX_t = \Delta X_t dt + \lambda X_t dt, \quad \lambda > 0.$$

Lower bound corresponds to unstable dimension of $\Delta + \lambda$.

Idea of the proof

Porous medium:

- Recall

$$dX_t = (\Delta(X_t^m) + \lambda X_t)\,dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^{(k)}. \tag{5}$$

- Semigroup of solutions is *not* differentiable in initial condition. Cannot proceed via linearization.
- Unstable manifold at 0:

$$\mathscr{M}^+(0,\omega) := \{u_0 \in X \mid \exists u : (-\infty,0] \to X, \text{ such that } \varphi(t;\theta_{-t}\omega)u(-t) = u_0 \\ \text{for all } t \geq 0 \text{ and } \|u(t)\|_{L^\infty(\mathscr{O})} \to 0 \text{ for } t \to -\infty\}.$$

  Lower bounds on the dimension of $\mathscr{M}^+(0,\omega)$ are enough.
- Two difficulties: Infinite time interval $(-\infty,0]$ and sufficiently distinct end-value u_0.

Idea of the proof

Infinite time interval

- As before $\mu_t := \sum_{k=1}^N f_k \beta_t^{(k)}$. The transformation $Y_t = e^{\mu_t - \lambda t} X_t$ leads to

$$\partial_t Y_t = e^{\mu_t - \lambda t} \Delta \left(e^{-\mu_t + \lambda t} Y_t \right)^m = e^{(m-1)\lambda t} e^{\mu_t} \Delta \left(e^{-\mu_t} Y_t \right)^m, \quad \text{on } (-\infty, 0] \times \mathscr{O}.$$

- Use a singular time-transformation: $G(t) = \frac{log(\delta t)}{\delta}$, $U_t := Y_{G(t)}$ satisfies

$$\partial_t U_t = e^{-\delta G(t)} e^{(m-1)\lambda G(t)} e^{\mu_{G(t)}} \Delta \left(e^{-\mu_{G(t)}} U_t \right)^m, \quad \text{on } (0, \tfrac{1}{\delta}] \times \mathscr{O}.$$

  Hence, the problem is reduced to a finite time interval.

Infinite dimensional random attractor

- Recall: Two difficulties: infinite time interval $(-\infty, 0]$ and sufficiently distinct end-value u_0.
- Sufficiently distinct end-values: Again consider

$$\partial_t U_t = e^{-\delta G(t)} e^{(m-1)\lambda G(t)} e^{\mu_{G(t)}} \Delta \left(e^{-\mu_{G(t)}} U_t \right)^m, \quad \text{on } (0, \tfrac{1}{\delta}] \times \mathscr{O}.$$

Due to finite speed of propagation: Support of solution stays disjoint if initially sufficiently far apart. In this case

$$\|U_t^1 - U_t^2\|_{L^1(\mathscr{O})} \geq \|U_t^1\|_{L^1(\mathscr{O})} + \|U_t^2\|_{L^1(\mathscr{O})} \geq e^{-Ct}(\|U_0^1\|_{L^1(\mathscr{O})} + \|U_0^2\|_{L^1(\mathscr{O})}).$$

E.g. choose $U_0^i = \chi_{B(\xi_i, \varepsilon)}$.
- In conclusion:

Theorem

Let $\mathscr{A}$ be a random set in X attracting all L^∞-bounded sets. Then,the fractal dimension $d_f(\mathscr{A}(\omega))$ is infinite for all $\omega \in \Omega$.

Thanks

Thanks!