Semi-discretization for stochastic scalar conservation laws with multiple rough fluxes

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[G., Souganidis; CMS, 2014], [G., Souganidis; CPAM, 2016],
[G., Perthame, Souganidis; SINUM, 2016].
Outline

1. Motivation
2. Deterministic case
3. Stochastic scalar conservation laws
Motivation
Motivation

- We will consider PDE driven by a 'rough' signal $z$ of the type

$$
du + \text{div}(A(x, u) \circ dz) = 0.
$$

If $A$ is a diagonal matrix this becomes

$$
du + \sum_{j=1}^{N} \partial_{x_j} A_j(x, u) \circ dz^j = 0
$$

- In particular, include $z = \beta$ Brownian motion.
- For example, stochastic Burgers’ equation

$$
du + \frac{1}{2} \partial_x u^2 \circ d\beta = 0.
$$
The motivation comes from two directions: Relation to Hamilton-Jacobi equations, mean-field games.

In the one-dimensional case: If $v$ solves the Hamilton-Jacobi equation

$$dv + A(\partial_x v, x) \circ d\beta = 0$$

then $u = \partial_x v$ solves

$$du + \partial_x A(u, x) \circ d\beta = 0.$$ 

But: The mathematical methods available for Hamilton-Jacobi equations (viscosity solutions) and scalar conservation laws (entropy solutions, kinetic methods) are very different.
Motivation

- Mean-field games going back to Lasry, Lions: Consider the SDE

\[
dX_t^i = \sigma \left( X_t^i, \frac{1}{L-1} \sum_{j \neq i} \delta_{X_t^j} \right) \circ d\beta_t \quad \text{in } \mathbb{R}^N
\]

for \( i = 1, \ldots, L \).

- Then the empirical law of \( X \) converges to a measure \( \pi_t \) with density \( m_t \) which evolves according to

\[
dm + \text{div}(\sigma^*(x, m) \circ d\beta) = 0.
\]

- Note that in general \( \sigma^* \) is not a diagonal matrix. We need the full generality of

\[
du + \text{div}(A(x, u) \circ d\beta) = 0.
\]
Deterministic case

1. Motivation

2. Deterministic case

3. Stochastic scalar conservation laws
Deterministic case

Consider

$$\partial_t u + \text{div} A(u) = \partial_t u + A'(u) \cdot \nabla u = 0.$$ 

The corresponding characteristic system reads

$$\frac{du}{dt} = 0, \quad \frac{dx}{dt} = A'(u)$$ 

$$u(0) = w, \quad x(0) = x.$$ 

Let $F^t(x, w)$ be the corresponding solution with initial condition $(x, w)$ at time $t = 0$. 

As long as $u$ is smooth ($u \in C^1$) we have

$$\text{graph}(u(t)) = F^t \text{graph}(u_0).$$ 

But: $F^t \text{graph}(u_0)$ gives a wrong (multivalued) solution once shocks appear.
Deterministic case

- From [Brenier, 1984]:
  - Need to ‘collapse’ multivalued solution into single-valued one.
  - Leads to Brenier’s transport-collapse scheme: For a timestep $\Delta t$ evolve via the characteristics, then collapse the possible multivalued solution to a single-valued one. Iterate.
Deterministic case

- Most efficiently described via the kinetic form of scalar conservation laws [Lions, Perthame, Tadmor; *JAMS*, 1994]
- Again consider
  \[ \partial_t u + \text{div} A(u) = 0. \]
- For simplicity say \( u_0 \geq 0 \), which implies \( u \geq 0 \).
- We consider the characteristic function
  \[ \chi(t, x, \xi) := 1_{[0, u(t, x)]}(\xi). \]

Elementary calculation (if \( u \) were smooth, i.e. no shocks):

\[
\partial_t \chi(t, x, \xi) := \delta_{\xi = u(t, x)} \partial_t u(t, x) = -\delta_{\xi = u(t, x)} \text{div} A(u) \\
= -\delta_{\xi = u(t, x)} A'(u) \cdot \nabla u = -\delta_{\xi = u(t, x)} A'(\xi) \cdot \nabla u \\
= -A'(\xi) \cdot \nabla 1_{[0, u(t, x)]}(\xi) = -A'(\xi) \cdot \nabla \chi(t, x, \xi).
\]
Deterministic case

- This is true up to shocks. The shocks introduce an error, the 'entropy dissipation measure' $m$:

$$
\partial_t \chi(t, x, \xi) + A'(\xi) \cdot \nabla \chi(t, x, \xi) = \partial_\xi m. \tag{1}
$$

- In the deterministic setting: $u$ is an entropy solution iff $\chi(t, x, \xi) := 1_{[0,u(t,x)]}(\xi)$ is a kinetic solution to (1).

- Advantage: (1) is a linear equation in $\chi$, at the expense of introducing the additional velocity variable $\xi$.

- Consider $\partial_\xi m$ as a Lagrange multiplier.
1 Motivation

2 Deterministic case

3 Stochastic scalar conservation laws
Stochastic scalar conservation laws

- **Aim**: Semi-discretization scheme for
  
  \[ du + \sum_{j=1}^{N} \partial_{x_j} A^i(x, u) \circ dz^j = 0. \]

- As before set
  
  \[ \chi(t, x, \xi) := 1_{[0, u(t, x)]}(\xi) \]

  which yields the kinetic form

  \[ d\chi + \sum_{j=1}^{N} \partial_{\xi} A^i(x, \xi) \partial_{x_j} \chi \circ dz^j + \sum_{j=1}^{N} \partial_{x_j} A^i(x, \xi) \partial_{\xi} \chi \circ dz^j = \partial_{\xi} m. \]
**Stochastic scalar conservation laws**

- Given time steps $0 = t_0 < t_1 < \cdots < t_K = T$.
- Iteratively:
  - First solve the linear "free-streaming" transport equation
    \[
    \partial_t f_{\Delta t} + \sum_{j=1}^N (\partial_\xi A^j)(x, \xi) \partial_{x_j} f_{\Delta t} \circ dz^j + \sum_{j=1}^N (\partial_{x_j} A^j)(x, \xi) \partial_\xi f_{\Delta t} \circ dz^j = 0
    \]
    on $[t_k, t_{k+1})$.
  - Then introduce a fast relaxation step, setting
    \[
    u_{\Delta t}(t, x) := \int f_{\Delta t}(t-, x, \eta) d\eta
    \]
    and
    \[
    f_{\Delta t}(t_{k+1}, x, \xi) := 1_{[0, u_{\Delta t}(t_{k+1}, x)]}(\xi).
    \]
- Take $u_{\Delta t}$ as an approximation of the pathwise entropy solution.
Deterministic case

- Literature: In the deterministic case, the convergence of this “transport collapse” scheme is known
  - [Brenier, SIAM, 1984]: Via compactness methods and $BV$-estimates.
  - [Vasseur, SIAM, 1999]: Via compactness methods and averaging Lemma.

- Known proofs do not allow to obtain a rate of convergence.
Consider the spatially homogeneous case

\[ du + \sum_{j=1}^{N} \partial_{x_j} A^j(u) \circ dz^j = 0, \]

where \( z \) is a continuous function.

**Theorem (G., Perthame, Souganidis; SINUM, 2016)**

Let \( u_0 \in (BV \cap L^\infty)(\mathbb{R}^N) \). Then

\[ \| u(t) - u_{\Delta t}(t) \|_{L^1} \leq C \sqrt{\Delta z}, \]

with a constant \( C \) given in terms of the data and

\[ \Delta z := \max_{k=0,\ldots,K-1} \sup_{t \in [t_k, t_{k+1}]} |z_t - z_{t_k}|. \]
Idea of the proof

- Semi-discretization scheme has a kinetic form:

\[
\partial_t f_{\Delta t} + \sum_{j=1}^{N} (\partial_\xi A^j(x, \xi) \partial_{x_j} f_{\Delta t} \circ dz^j = \sum_k \delta(t - t_k)(\mathcal{M} f_{\Delta t} - f_{\Delta t}) =: \partial_\xi m_{\Delta t},
\]

where

\[
\mathcal{M} f_{\Delta t} := 1_{[0, u_{\Delta t}(t,x)]}(\xi) - \int_{f_{\Delta t}(t,x,\eta)d\eta}(\xi).
\]

- Use this to derive estimate for

\[
\int |\chi(t) - f_{\Delta t}(t)| d\xi dx.
\]

- Obstacle: While

\[
\partial_\xi 1_{[0, u(t,x)]}(\xi) = \delta(\xi) - \delta(u(t,x) - \xi) \leq \delta(\xi)
\]

we only have

\[
\partial_\xi f_{\Delta t} = \partial_\xi (f_{\Delta t}(t_k, x - A'(\xi)(z_t - z_{t_k}), \xi)) \leq \delta(\xi) + D_x f_{\Delta t}(x - A'(\xi)(z_t - z_{t_k}), \xi, t_k) \cdot A''(\xi)(z_t - z_{t_k})
\]

- Compensate blow-up of \(D_x f_{\Delta t}\) by convergence of \(|z_t - z_{t_k}| \leq \Delta z\) via BV estimates.
Consider the spatially inhomogeneous case

\[ du + \sum_{j=1}^{N} \partial_{x_j} A^i(x, u) \circ dz^j = 0, \]

where \( z \) is an \( \alpha \)-Hölder rough path, \( A \) sufficiently smooth.

**Theorem** (G., Perthame, Souganidis; SINUM, 2016)

*Let* \( u_0 \in (L^1 \cap L^2)(\mathbb{R}^N) \). *Then, for* \( \Delta t \to 0 \),

\[ u_{\Delta t} \to u \quad \text{in} \quad L^1([0, T] \times \mathbb{R}^N). \]
Idea of the proof

- Principle obstacle: No BV estimates, no averaging techniques.
- Still have a kinetic form

\[ \partial_t f_{\Delta t} + \sum_{j=1}^{N} (\partial_\xi A^j)(x, \xi) \partial_{x_j} f_{\Delta t} \circ dz^j + \sum_{j=1}^{N} (\partial_{x_j} A^j)(x, \xi) \partial_\xi f_{\Delta t} \circ dz^j \]

\[ = \sum_{k} \delta(t - t_k) (\mathcal{M} f_{\Delta t} - f_{\Delta t}) =: \partial_\xi m_{\Delta t}, \quad (\star) \]

where

\[ \mathcal{M} f_{\Delta t} := 1_{[0,u_{\Delta t}(t,x)]}(\xi) = 1_{[0,\int f_{\Delta t}(t,x,\eta)d\eta]}(\xi). \]

- Derive stable $L^1$ and $L^2$ bounds for $f_{\Delta t}$. Then pass to weak limit in $(\star)$. Leads to a generalized entropy solution.
- From [G., Souganidis, CMS, 2015] we know: generalized entropy solutions are entropy solutions.
- Deduce strong convergence by proving uniform tightness for $f_{\Delta t}$. 
Thanks!