Semi-discretization for stochastic scalar conservation laws with multiple rough fluxes

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[G., Souganidis; CMS, 2014], [G., Souganidis; CPAM, 2016],
[G., Perthame, Souganidis; SINUM, 2016].

Outline

Motivation

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- We will consider PDE driven by a 'rough' signal z of the type

$$du + \mathrm{div}(A(x,u) \circ dz) = 0.$$

  If A is a diagonal matrix this becomes

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(x,u) \circ dz^j = 0$$

- In particular, include $z = \beta$ Brownian motion.
- For example, stochastic Burgers' equation

$$du + \frac{1}{2}\partial_x u^2 \circ d\beta = 0.$$

Motivation

- The motivation comes from two directions: Relation to Hamilton-Jacobi equations, mean-field games.
- In the one-dimensional case: If v solves the Hamilton-Jacobi equation

$$dv + A(\partial_x v, x) \circ d\beta = 0$$

then $u = \partial_x v$ solves

$$du + \partial_x A(u, x) \circ d\beta = 0.$$

- But: The mathematical methods available for Hamilton-Jacobi equations (viscosity solutions) and scalar conservation laws (entropy solutions, kinetic methods) are very different.

Motivation

- Mean-field games going back to Lasry, Lions: Consider the SDE

$$dX_t^i = \sigma\left(X_t^i, \frac{1}{L-1}\sum_{j\neq i}\delta_{X_t^j}\right)\circ d\beta_t \quad \text{in } \mathbb{R}^N$$

  for $i = 1, \ldots, L$.
- Then the empirical law of X converges to a measure π_t with density m_t which evolves according to

$$dm + \operatorname{div}(\sigma^*(x,m)\circ d\beta) = 0.$$

- Note that in general σ^* is not a diagonal matrix. We need the full generality of

$$du + \operatorname{div}(A(x,u)\circ d\beta) = 0.$$

Deterministic case

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Deterministic case

- Consider
$$\partial_t u + \mathrm{div} A(u) = \partial_t u + A'(u) \cdot \nabla u = 0.$$
- The corresponding characteristic system reads
$$\frac{du}{dt} = 0, \quad \frac{dx}{dt} = A'(u)$$
$$u(0) = w, \quad x(0) = x.$$
- Let $F^t(x, w)$ be the corresponding solution with initial condition (x, w) at time $t = 0$.
- As long as u is smooth $(u \in C^1)$ we have
$$graph(u(t)) = F^t graph(u_0).$$
- But: $F^t graph(u_0)$ gives a wrong (multivalued) solution once shocks appear.

Deterministic case

- From [Brenier, 1984]:

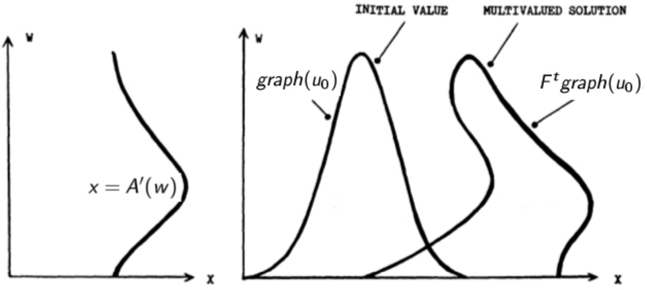

FIG. 1. *Method of characteristics.*

- Need to 'collapse' multivalued solution into single-valued one.
- Leads to Brenier's transport-collapse scheme: For a timestep Δt evolve via the characteristics, then collapse the possible multivalued solution to a single-valued one. Iterate.

Deterministic case

- Most efficiently described via the kinetic form of scalar conservation laws [Lions, Perthame, Tadmor; *JAMS*, 1994]
- Again consider
$$\partial_t u + \mathrm{div} A(u) = 0.$$
- For simplicity say $u_0 \geq 0$, which implies $u \geq 0$.
- We consider the characteristic function
$$\chi(t,x,\xi) := 1_{[0,u(t,x)]}(\xi).$$

Elementary calculation (if u were smooth, i.e. no shocks):

$$\begin{aligned}\partial_t \chi(t,x,\xi) : &= \delta_{\xi=u(t,x)} \partial_t u(t,x) = -\delta_{\xi=u(t,x)} \mathrm{div} A(u) \\ &= -\delta_{\xi=u(t,x)} A'(u) \cdot \nabla u = -\delta_{\xi=u(t,x)} A'(\xi) \cdot \nabla u \\ &= -A'(\xi) \cdot \nabla 1_{[0,u(t,x)]}(\xi) = -A'(\xi) \cdot \nabla \chi(t,x,\xi).\end{aligned}$$

Deterministic case

- This is true up to shocks. The shocks introduce an error, the '*entropy dissipation measure*' m:

$$\partial_t \chi(t,x,\xi) + A'(\xi) \cdot \nabla \chi(t,x,\xi) = \partial_\xi m. \tag{1}$$

- In the deterministic setting: u is an entropy solution iff $\chi(t,x,\xi) := 1_{[0,u(t,x)]}(\xi)$ is a kinetic solution to (1).
- Advantage: (1) is a linear equation in χ, at the expense of introducing the additional velocity variable ξ.
- Consider $\partial_\xi m$ as a Lagrange multiplier.

Stochastic scalar conservation laws

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Stochastic scalar conservation laws

- Aim: Semi-discretization scheme for

$$du + \sum_{j=1}^{N} \partial_{x_j} A^j(x,u) \circ dz^j = 0.$$

- As before set

$$\chi(t,x,\xi) := 1_{[0,u(t,x)]}(\xi)$$

which yields the kinetic form

$$d\chi + \sum_{j=1}^{N} \partial_\xi A^j(x,\xi) \partial_{x_j} \chi \circ dz^j + \sum_{j=1}^{N} \partial_{x_j} A^j(x,\xi) \partial_\xi \chi \circ dz^j = \partial_\xi m.$$

Stochastic scalar conservation laws

- Given time steps $0 = t_0 < t_1 < \cdots < t_K = T$.
- Iteratively:
  - First solve the linear "free-streaming" transport equation

$$\partial_t f_{\Delta t} + \sum_{j=1}^{N} (\partial_\xi A^j)(x,\xi)\partial_{x_j} f_{\Delta t} \circ dz^j + \sum_{j=1}^{N} (\partial_{x_j} A^j)(x,\xi)\partial_\xi f_{\Delta t} \circ dz^j = 0$$

    on $[t_k, t_{k+1})$.
  - Then introduce a fast relaxation step, setting

$$u_{\Delta t}(t,x) := \int f_{\Delta t}(t-,x,\eta)d\eta$$

    and

$$f_{\Delta t}(t_{k+1},x,\xi) := 1_{[0,u_{\Delta t}(t_{k+1},x)]}(\xi).$$

- Take $u_{\Delta t}$ as an approximation of the pathwise entropy solution.

Deterministic case

- Literature: In the deterministic case, the convergence of this "transport collapse" scheme is known
  - [Brenier, SIAM, 1984]: Via compactness methods and BV-estimates.
  - [Vasseur, SIAM, 1999]: Via compactness methods and averaging Lemma.
- Known proofs do *not* allow to obtain a rate of convergence.

Homogeneous stochastic scalar conservation laws

- Consider the spatially homogeneous case

$$du + \sum_{j=1}^{N} \partial_{x_j} A^j(u) \circ dz^j = 0,$$

where z is a continuous function.

Theorem (G., Perthame, Souganidis; SINUM, 2016)

Let $u_0 \in (BV \cap L^\infty)(\mathbb{R}^N)$. Then

$$\|u(t) - u_{\Delta t}(t)\|_{L^1} \leq C\sqrt{\Delta z},$$

with a constant C given in terms of the data and

$$\Delta z := \max_{k=0,\dots,K-1} \sup_{t \in [t_k, t_{k+1}]} |z_t - z_{t_k}|.$$

Idea of the proof

- Semi-discretization scheme has a kinetic form:

$$\partial_t f_{\Delta t} + \sum_{j=1}^{N} (\partial_\xi A^j)(x,\xi)\partial_{x_j} f_{\Delta t} \circ dz^j = \sum_k \delta(t-t_k)(\mathscr{M} f_{\Delta t} - f_{\Delta t}) =: \partial_\xi m_{\Delta t},$$

  where

$$\mathscr{M} f_{\Delta t} := 1_{[0,u_{\Delta t}(t,x)]}(\xi) = 1_{[0,\int f_{\Delta t}(t,x,\eta)d\eta]}(\xi).$$

- Use this to derive estimate for

$$\int |\chi(t) - f_{\Delta t}(t)| d\xi\, dx.$$

- Obstacle: While

$$\partial_\xi 1_{[0,u(t,x)]}(\xi) = \delta(\xi) - \delta(u(t,x)-\xi) \leq \delta(\xi)$$

  we only have

$$\begin{aligned}\partial_\xi f_{\Delta t} &= \partial_\xi (f_{\Delta t}(t_k, x - A'(\xi)(z_t - z_{t_k}), \xi)) \\ &\leq \delta(\xi) + D_x f_{\Delta t}(x - A'(\xi)(z_t - z_{t_k}), \xi, t_k) \cdot A''(\xi)(z_t - z_{t_k})\end{aligned}$$

- Compensate blow-up of $D_x f_{\Delta t}$ by convergence of $|z_t - z_{t_k}| \leq \Delta z$ via BV estimates.

Inhomogeneous stochastic scalar conservation laws

- Consider the spatially inhomogeneous case

$$du + \sum_{j=1}^{N} \partial_{x_j} A^j(x,u) \circ dz^j = 0,$$

where z is an α-Hölder rough path, A sufficiently smooth.

Theorem (G., Perthame, Souganidis; SINUM, 2016)

Let $u_0 \in (L^1 \cap L^2)(\mathbb{R}^N)$. Then, for $\Delta t \to 0$,

$$u_{\Delta t} \to u \quad \text{in } L^1([0,T] \times \mathbb{R}^N).$$

Idea of the proof

- Principle obstacle: No BV estimates, no averaging techniques.
- Still have a kinetic form

$$\partial_t f_{\Delta t} + \sum_{j=1}^{N} (\partial_\xi A^j)(x,\xi)\partial_{x_j} f_{\Delta t} \circ dz^j + \sum_{j=1}^{N} (\partial_{x_j} A^j)(x,\xi)\partial_\xi f_{\Delta t} \circ dz^j$$
$$= \sum_k \delta(t-t_k)(\mathscr{M} f_{\Delta t} - f_{\Delta t}) =: \partial_\xi m_{\Delta t}, \qquad (\star)$$

where

$$\mathscr{M} f_{\Delta t} := 1_{[0,u_{\Delta t}(t,x)]}(\xi) = 1_{[0,\int f_{\Delta t}(t,x,\eta)d\eta]}(\xi).$$

- Derive stable L^1 and L^2 bounds for $f_{\Delta t}$. Then pass to weak limit in $(\star)$. Leads to a generalized entropy solution.
- From [G., Souganidis, CMS, 2015] we know: generalized entropy solutions are entropy solutions.
- Deduce strong convergence by proving uniform tightness for $f_{\Delta t}$.

Thanks

Thanks!