Synchronization by noise for order-preserving random dynamical systems

Benjamin Gess
Universität Bielefeld

Equadiff 2015
Lyon, July 2015

joint work with: Franco Flandoli, Michael Scheutzow
[arXiv:1503.08737]

Outline

1. Introduction
2. Obstacles: Application to SPDE - stochastic porous medium equations
3. Weak synchronization for order-preserving RDS

Introduction

Introduction
- one dimensional case -

Synchronization by noise

- Consider the one-dimensional SDE

$$dX_t = b(X_t)dt + dW_t \quad \text{in } \mathbb{R}. \tag{*}$$

- The inclusion of noise may simplify the long-time dynamics, i.e. while

$$dX_t = b(X_t)dt$$

may not be globally stable, the long-time behavior of $(*)$ may be trivial.

- Roughly speaking: Synchronization by noise means that the random attractor consists of a single random point, i.e.

$$A(\omega) = \{a(\omega)\}, \quad \mathbb{P}\text{-a.s.}$$

- In particular: If synchronization occurs, then each two trajectories converge to each other in probability:

$$|X_t^x - X_t^y| \to 0 \quad \text{for } t \to \infty$$

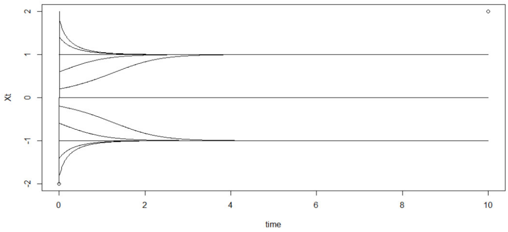
in probability.

Model example

- Deterministic case ($\sigma = 0$, $d = 1$):

$$dX_t = (X_t - X_t^3)dt$$

- Attractor is given by closed unit ball: $A = \bar{B}_1(0) = [-1,1]$.
- Point attractor is given by $S^{d-1} \cup \{0\} = \{\pm 1, 0\}$.
- Simulation:

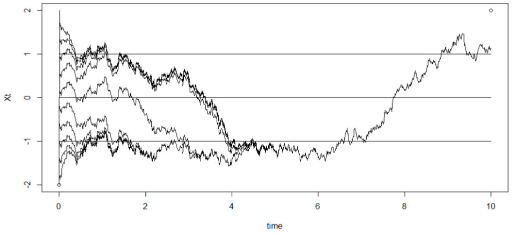


Model example

- Additive noise ($\sigma > 0$):

$$dX_t = (X_t - X_t^3)dt + \sigma dW_t$$

- Synchronization occurs: $A(\omega) = \{a(\omega)\}$ a.s.. In particular $|X_t^x - X_t^y| \to 0$ for $t \to \infty$ in probability.
- Simulation:



Random attractors and synchronization

Let φ be a RDS on some Polish space (E, d).

Definition

A *weak random attractor* is a random compact set $A(\omega)$ such that

1. (invariance): $\varphi_t(\omega)A(\omega) = A(\theta_t\omega)$, a.s. for all $t \geq 0$.
2. (attraction):
$$d(\varphi_t(\omega)B, A(\theta_t\omega)) \to 0 \quad \text{for } t \to \infty$$
in probability, for each compact set B.

If we replace compact sets B by points, then A is called a *weak point attractor*.

Definition

We say that *synchronization* occurs if the weak random attractor is a singleton

$$A(\omega) = \{a(\omega)\} \quad \text{a.s.}.$$

We say that *weak synchronization* occurs if there is a singleton weak point attractor.

Proof for one dimension

Theorem (Crauel, Flandoli; JDDE, 1998)

The random attractor corresponding to

$$dX_t = (X_t - X_t^3)dt + \sigma dW_t, \quad \sigma > 0,$$

consists of a single random point, i.e. synchronization holds.

Ingredients of the proof :

- Associated RDS $\varphi_t(\omega, x) := X_t^x(\omega)$ and Markovian semigroup $P_t f(x) := \mathbb{E} f(X_t^x)$. P_t has a unique invariant measure μ and there is a random attractor $\mathscr{A}(\omega)$ for φ.
- Since $\mathscr{A}(\omega)$ is compact a.s. one may choose maximal/minimal elements $a_+(\omega)$, $a_-(\omega) \in \mathscr{A}(\omega)$ such that

$$\mathscr{A}(\omega) \subseteq [a_-(\omega), a_+(\omega)].$$

Proof for one dimension

- Since φ is order-preserving we have

$$\begin{aligned}\varphi_t(\omega)\mathscr{A}(\omega) &\subseteq \varphi_t(\omega)[a_-(\omega),a_+(\omega)]\\ &\subseteq [\varphi_t(\omega)a_-(\omega),\varphi_t(\omega)a_+(\omega)].\end{aligned}$$

- By invariance of $\mathscr{A}(\omega)$ one obtains

$$\varphi_t(\omega)a_\pm(\omega)=a_\pm(\theta_t\omega).$$

- Hence,

$$\mathscr{L}(a_+),\mathscr{L}(a_-)=\mu.$$

- This implies $a_+(\omega)=a_-(\omega)$ a.s. and thus

$$\mathscr{A}(\omega)=\{a_-(\omega)\}\text{ a.s..}$$

We have used two main ingredients:

- φ is order-preserving and P_t has a unique invariant measure
- *admissibility*: compact sets K in $\mathbb{R}$ are contained in intervals,

$$K\subseteq[f,g].$$

Obstacles: Application to SPDE

Obstacles: Application to SPDE
- stochastic porous medium equations -

Application to SPDE

- The same arguments as for the one dimensional case can be used on partially ordered Banach spaces $(E, d, \leq)$.
  E.g. [Caraballo, Crauel, Langa, Robinson; PAMS, 2006]: Synchronization for stochastic Chaffee-Infante for $d = 1$.
- Admissibility becomes a restrictive assumption. For example, L^p spaces are not admissible.
- Approach only useful in regular cases, i.e. if we can work with solutions that are continuous in space $E = C^0(\mathscr{O})$.
- Obstacle: Uniqueness of invariant measures requires a lot of noise, which causes irregular solutions.

The stochastic porous medium equation

- As a model example consider the stochastic porous medium equation

$$dX_t = \left(\Delta X_t^{[m]} + X_t\right) dt + dW_t,$$

with zero Dirichlet boundary conditions on a bounded, smooth domain $\mathscr{O} \subseteq \mathbb{R}^d$, $d \leq 3$, $m > 1$.

- There is an associated RDS φ on $E = H^{-1} := (H_0^1)^*$.
- Assuming some non-degeneracy for W_t: The Markov semigroup $P_t f(x) := \mathbb{E} f(\varphi_t(\cdot, x))$ is strongly mixing with invariant measure μ, that is,

$$\mathscr{L}(\varphi_t(\cdot, x)) \rightharpoonup \mu, \quad \text{for } t \to \infty$$

and μ is concentrated on $L^{m+1} \subseteq H^{-1}$.

- Better estimates for μ than for $\mathscr{A}$ $\to$ require a different method based on μ rather than $\mathscr{A}$.

Application to SPDE

Recall:

- *admissibility*: For each compact set $K \subseteq E$ there is an interval such that $K \subseteq int([f,g])$.
- *normality:* If $0 \leq f \leq g$ then $\|f\|_E \leq C\|g\|_E$.
- φ is order-preserving if for all $x \leq y$ we have

$$\varphi_t(\omega,x) \leq \varphi_t(\omega,y) \quad \forall t \geq 0, \omega \in \Omega.$$

Theorem (Chueshov, Scheutzow; DS, 2004)

Assume that φ is strongly mixing, that is, $\mathscr{L}(\varphi_t(\cdot,x)) \rightharpoonup \mu$ weakly for all $x \in E$. Further assume that $\leq$ is admissible and normal. Then there is a weak attractor $\mathscr{A}(\omega)$ consisting of a single random point, i.e. synchronization holds.

Two problems:

1. Assumptions of admissibility and normality
2. Artificial restriction to Banach spaces.

The stochastic porous medium equation

Admissibility:

- Recall: "$\leq$" is admissible if for each compact set $K \subseteq E$ there is an interval $[f, g]$ such that $K \subseteq int([f, g])$.

Normality:

- Recall: If "$\leq$" is normal then $\text{diam}([f, g]) \leq C|f - g|$ for all intervals $[f, g]$.

Observation:

- Admissibility and normality are conflicting assumptions.

The stochastic porous medium equation

Recall

$$dX_t = \left(\Delta X_t^{[m]} + X_t\right) dt + dW_t \tag{SPME}$$

The stochastic porous medium equation satisfies a comparison principle:

- For two distributions $x, y \in H^{-1}$ we can introduce the (standard) partial order "$\leq$" on H^{-1} by $x \leq y$ iff
$$y(v) \geq x(v)$$
for all nonnegative $v \in C_c^\infty$.
- In particular: For $x, y \in L^2 \subseteq H^{-1}$ we have $x \leq y$ iff
$$x(\xi) \leq y(\xi) \text{ for a.e. } \xi \in \mathscr{O}.$$
- "$\leq$" is preserved by φ.
- Does the partial order on H^{-1} satisfy admissibility? No. Even $L^{m+1} = \mathrm{supp}(\mu)$ is not admissible.

The stochastic porous medium equation

- Key idea: Introduce non-standard partial order on H^{-1} with larger intervals $[f,g]$.
- For $x,y \in H^{-1}$ let $x \preceq y$ iff

$$(-\Delta)^{-1}x \leq (-\Delta)^{-1}y.$$

- Then $\varphi_t(\omega,x) := X_t^x(\omega)$ is $\preceq$-order preserving.
- The partial order $\preceq$ is admissible if $d = 1$.
- idea: It should be sufficient to have: For all $\varepsilon > 0$ there is an interval $[f,g]_\preceq$ such that

$$\mu([f,g]_\preceq) \geq 1-\varepsilon.$$

  This is true for all $d \leq 4$.

The stochastic porous medium equation

- But: There are intervals $[f,g]_{\preceq}$ such that

$$\mathrm{diam}([f,g]_{\preceq}) = \infty.$$

  In particular $\preceq$ is not normal.
- Conclusion: Need a general theory for partial orders that are not normal.

Order-preserving RDS

Weak synchronization for order-preserving RDS

Order-preserving RDS

- Let (E,d) be a Polish space with closed partial order "$\leq$"
- Let φ be an order-preserving RDS and φ be strongly mixing with invariant measure μ, i.e. for all $x \in E$

$$\mathscr{L}(\varphi_t(\cdot,x)) \rightharpoonup \mu, \quad \text{for } t \to \infty.$$

Theorem

Assume that μ is concentrated on intervals, i.e. for all $\varepsilon > 0$ there exists an interval $[f,g] \subseteq E$ such that

$$\mu([f,g]) \geq 1-\varepsilon.$$

Then, the support of the statistical equilibrium is given by a single random point $supp \mu_\omega = \{a(\omega)\}$ and

$$A(\omega) := \{a(\omega)\}$$

is a singleton weak point attractor, i.e. weak synchronization holds.

Order-preserving RDS

ideas of the proof:

Proposition

Let φ be strongly mixing, order-preserving and $f \leq g$. Then, for all $x, y \in [f, g]$:

$$d(\varphi_t(\omega, x), \varphi_t(\omega, y)) \to 0 \quad \text{for } t \to \infty$$

in probability. In other words: φ is weakly asymptotically stable on each interval $[f, g]$.

- Recall: If "$\leq$" is normal then $\operatorname{diam}([f, g]) \leq Cd(f, g)$ for all intervals $[f, g]$.

Order-preserving RDS

Definition

A partial order is normal if there is a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\downarrow 0} h(t) = 0$ and

$$\operatorname{diam}([f,g]) \le h(d(f,g))$$

for each $f \le g$.

Proposition

Let $K \subseteq E$ compact. Then (K,d) is a Polish space with normal partial order $\le$.

Order-preserving RDS

- Hence, for all $f \le g$ and K compact:

$$\operatorname{diam}(\varphi_t(\cdot,[f,g]) \cap K) \to 0$$

  in probability.
- Combining this with

$$\mu([f,g]) \ge 1-\varepsilon$$

  and strong mixing, i.e.

$$\mathscr{L}(\varphi_t(\cdot,f)), \mathscr{L}(\varphi_t(\cdot,g)) \rightharpoonup \mu$$

  we obtain that

$$\mu_\omega(\{a(\omega)\}) = 1 \quad \mathbb{P}\text{-a.s.}$$

  for some random variable a.

Order-preserving RDS

In particular we obtain:

Theorem

Weak synchronization holds for

$$dX_t = \left(\Delta X_t^{[m]} + X_t\right) dt + dW_t,$$

with $d \leq 4$, $m > 1$.

Thanks

Thanks!