Fluctuations in non-equilibrium and stochastic PDE

Benjamin Gess
Universität Bielefeld
& Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig

Graduate School "Mathematics of random systems: Analysis, modelling and algorithms", Imperial College London, September 2021

joint work with Ben Fehrman [https://arxiv.org/abs/1910.11860]

Max-Planck-Institut für Mathematik in den Naturwissenschaften

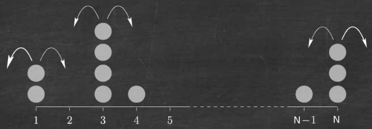
Universität Bielefeld

Content

Conservative SPDE as fluctuating continuum models

Two ways to the LDP, the skeleton equation

The zero range process
(could also consider simple exclusion, independent particles).

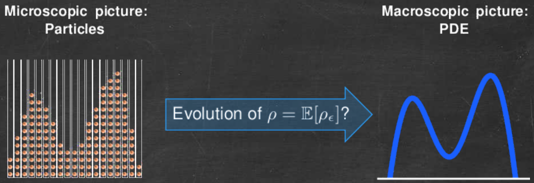


- State space $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N}$, i.e. configurations $\eta : \mathbb{T}_N \to \mathbb{N}_0$: System in state η if container k contains $\eta(k)$ particles.
- Local jump rate function $g : \mathbb{N}_0 \to \mathbb{R}_0^+$.
- Translation invariant, asymmetric, zero mean transition probability

$$p(k, l) = p(k - l), \quad \sum_k kp(k) = 0.$$

- Markov jump process $\eta(t)$ on $\mathbb{M}_N$.
- $\eta(k, t) = \text{number of particles in box } k \text{ at time } t.$

- **Hydrodynamic limit?** Multi-scale dynamics



- Empirical density field: $\mu^N(x,t) := \frac{1}{N}\sum_k \delta_{\frac{k}{N}}(x)\eta(k, tN^2)$.
- [Hydrodynamic limit - Ferrari, Presutti, Vares; 1987]

$$\mu^N(t) \rightharpoonup^* \bar{\rho}(t)\,dx$$

with

$$\partial_t \bar{\rho} = \frac{1}{2}\partial_{xx}\Phi(\bar{\rho})$$

with Φ the mean local jump rate $\Phi(\rho) = \mathbb{E}_{\nu_\rho}[g(\eta(0))]$.
- Loss of information:
  - Error: $\mu^N = \bar{\rho} + o(1)$
  - Fluctuations, rare events?

Rate of convergence?

- Higher order expansion / fluctuation correction: Ansatz

$$\mu^N = \bar{\rho} + \frac{1}{N^{\frac{1}{2}}} Y^1 + \frac{1}{N} Y^2 + \ldots$$

  What are Y^i?

- [Central limit fluctuations in non-equilibrium - Ferrari, Presutti, Vares; 1988]: Fluctuation density fields

$$\begin{aligned} Y^{1,N}(x,t) &= N^{\frac{1}{2}}(\mu^N(x,t) - \mathbb{E}\mu^N(x,t)) \\ &\approx N^{\frac{1}{2}}(\mu^N(x,t) - \bar{\rho}) \end{aligned}$$

  Then,

$$\mathcal{L}(Y^{1,N}) \rightharpoonup^* \mathcal{L}(Y^1) \text{ for } N \to \infty$$

  with Y^1 the (Gaussian) solution to

$$dY^1(x,t) = \partial_{xx}(\Phi'(\bar{\rho}(x,t)) Y^1(x,t))\, dt + \partial_x(\Phi^{\frac{1}{2}}(\bar{\rho}(x,t)) dW(t))$$

  with dW space-time white noise.

- Therefore,

$$\mu^N = \overbrace{\bar{\rho} + \frac{1}{N^{\frac{1}{2}}} Y^1}^{=:\bar{\rho}^N} + o(\frac{1}{N^{\frac{1}{2}}}).$$

Rare events?

- [Large deviation principle, Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995]: Let now ρ_0 constant. Then, informally,

$$\mathbb{P}[\mu^N \approx \rho\, dx] \approx \exp\{-N\, I_0(\rho dx)\},$$

with rate function

$$I_0(\rho dx) = \inf\left\{\int_0^T \int_{\mathbb{T}} |g|^2 dx ds : \ g \in L^2_{t,x},\ \underbrace{\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho) g)}_{\text{"skeleton equation"}}\right\}$$

- Note: This does **not** coincide with the rate function of the linearly corrected continuum model $\bar{\rho}^N$.

Question: Fluctuation correction implying higher order of approximation and correct rare events?

Ansatz: Langevin dynamics (nonlinear!)

$$\partial_t \rho^N = \partial_{xx}\left(\Phi(\rho^N)\right) + \frac{1}{N^{\frac{1}{2}}}\partial_x\left(\Phi^{\frac{1}{2}}(\rho^N)dW_t\right).$$

Model case: Dean-Kawasaki, independent particles, $\Phi(\rho) = \rho$, i.e.

$$\partial_t \rho^N = \partial_{xx}\rho^N + \frac{1}{N^{\frac{1}{2}}}\partial_x\left((\rho^N)^{\frac{1}{2}}dW_t\right).$$

Informal justification:

1. Physics: Fluctuation-dissipation relation, "fluctuating hydrodynamics"
2. Law of large numbers, Central limit fluctuations (improved order of approximation)
   & *correct large deviations*

Informally, correct rare events:

- Recall

$$\partial_t \rho^N = \partial_{xx}\left(\Phi(\rho^N)\right) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho^N)}dW_t\right).$$

- Informally applying the contraction principle to the solution map

$$F : \frac{1}{\sqrt{N}}dW \mapsto \rho$$

  yields as a rate function

$$I(\rho) = \inf\{I_{dW}(g) : F(g) = \rho\}.$$

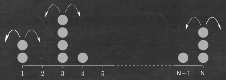
- Schilder's theorem for Brownian sheet suggests

$$I_{dW}(g) = \int_0^T \int_{\mathbb{T}} |g|^2 \, dxdt.$$

- Get

$$I(\rho) = \inf\left\{\int_0^T \int_{\mathbb{T}} |g|^2 \, dxdt : \partial_t \rho = \partial_{xx}\left(\Phi(\rho)\right) + \partial_x\left(\sqrt{\Phi(\rho)}g\right)\right\}.$$

- Obstacle
$$\partial_t \rho = \partial_{xx}\left(\Phi(\rho)\right) + \frac{1}{N^{\frac{1}{2}}}\partial_x\left(\Phi^{\frac{1}{2}}(\rho)dW_t\right)$$
  1. not well-posed, supercritical -> no regularity structures
  2. Renormalization? Does renormalization appear in rate function? E.g. compare $\Phi^4_{2/3}$ [Hairer, Weber; 2014].
- Decorrelation length of discrete system $= \frac{1}{N}$.



$$\partial_t \rho^N = \partial_{xx}\left(\Phi(\rho^N)\right) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho^N)}dW^N_t\right)$$

  where W^N has correlation length $\frac{1}{N}$.
- Ansatz: joint limit "small noise, ultraviolet cutoff"
$$\partial_t \rho^{N,K} = \partial_{xx}\left(\Phi(\rho^{N,K})\right) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho^{N,K})}\circ dW^K_t\right)$$

  where W^K has correlation length $\frac{1}{K}$.
- Gives the correct rate function for $\frac{1}{N} << \frac{1}{K}$.

Note: This is a particular case in which the link between *Macroscopic fluctuation theory* [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015] and *fluctuating hydrodynamics* [Landau-Lifshitz 1973, Spohn 1991] can be made rigorous.

Two ways to the LDP, the skeleton equation

Conservative SPDE as fluctuating continuum models

Two ways to the LDP, the skeleton equation

- In the following concentrate on the case
$$\Phi(\rho) = \rho^m, \quad m \geq 1.$$
- We consider stochastic PDE of the type
$$\partial_t \rho^{N,K} = \Delta\left((\rho^{N,K})^m\right) + \frac{1}{\sqrt{N}} \mathrm{div}\left((\rho^{N,K})^{\frac{m}{2}} \circ dW_t^K\right), \qquad (*)$$
on $\mathbb{T}^d \times (0,\infty)$, where $W^K = \sum_{k=1}^K e_k \beta^k$.
- Pathwise well-posedness of $(*)$: [Lions, Souganidis; 1998ff], [Lions, Perthame, Souganidis; 2013], [Lions, Perthame, Souganidis; 2014], [G., Souganidis; 2014], [G., Souganidis; 2015], [G., Fehrman; 2017], [Dareiotis, G.; 2019], [Fehrman, G.; 2021].

Two ways to the LDP:

1. Γ-convergence of the rate functional: $N \uparrow \infty$ yields LDP for $(*)$ with rate function
$$I^K(\rho) = \inf\left\{\int_0^T \int_{\mathbb{T}^d} |g|^2\, dxdt : \partial_t \rho = \partial_{xx}\rho^m + \partial_x\left(\rho^{\frac{m}{2}} P^K g\right)\right\}.$$
Then consider $K \uparrow \infty$.
2. Joint scaling: Weak convergence approach to LDP ($\frac{1}{N} << \frac{1}{K}$).

What do we need to show?E.g. Γ-convergence of the rate function

$$I^K(\rho) = \inf\left\{\int_0^T\int_{\mathbb{T}^d} |g|^2\, dxdt : \partial_t\rho = \partial_{xx}\rho^m + \partial_x\left(\rho^{\frac{m}{2}} P^K g\right)\right\}.$$

- Let $\rho^K \to \rho$ need to show

$$I(\rho) \leq \liminf_K I^K(\rho^K).$$

- Choose g^K such that

$$I^K(\rho^K) = \int_0^T\int_{\mathbb{T}^d} |g^K|^2\, dxdt \quad \text{and} \quad \partial_t\rho^K = \partial_{xx}\left(\rho^K\right)^m + \partial_x\left((\rho^K)^{\frac{m}{2}}\underbrace{P^K g^K}_{\rightharpoonup g}\right).$$

- Then $g^K \rightharpoonup g$ in $L^2_{t,x}$. Need to show $\rho^K \to \rho$ with

$$\partial_t\rho = \partial_{xx}\rho^m + \partial_x\left(\rho^{\frac{m}{2}} g\right).$$

- Both approaches crucially depend on understanding the skeleton PDE.
- The skeleton equation

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}\left(\rho^{\frac{m}{2}} g(t,x)\right) \qquad (*)$$
$$\rho(0,x) = \rho_0(x),$$

  with $g \in L^2_{t,x}$?
- This leads to the key problem

Problem

1. *Existence and uniqueness of solutions to* $(*)$.
2. *Stability of solutions: Let* $g^n \rightharpoonup g$ *in* $L^2_{t,x}$ *with corresponding solutions* ρ^n, ρ. *Then*
$$\rho^n \to \rho$$
in $L^\infty_t L^1_x$.

- Difficulty: Stable a-priori bound? L^p framework does not work.
- Do we expect non-concentration of mass / well-posedness?

Scaling and criticality of the skeleton equation

- We consider

$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d$$

  with $g \in L^q(\mathbb{R}_{+,t}; L^p(\mathbb{R}^d_x; \mathbb{R}^d_x))$ and $\rho_0 \in L^r(\mathbb{R}^d_x)$.
- Via rescaling ("zooming in"):
  - $p = q = 2$ is critical.
  - $r = 1$ is critical, $r > 1$ is supercritical.

Apriori-bounds and energy space

- Consider
$$\partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d \qquad (*)$$
with $g \in L^2(\mathbb{R}_{+,t}; L^2(\mathbb{R}^d_x; \mathbb{R}^d_x))$.
- L^1 estimate only gives
$$\int_{\mathbb{T}^d} \rho(t,x)dx = \int_{\mathbb{T}^d} \rho_0(x)dx.$$
- Use entropy-entropy dissipation: Evolution of entropy given by $\int_{\mathbb{T}^d} \log(\rho)\rho$. Informally gives
$$\int_{\mathbb{T}^d} \log(\rho)\rho \, dx\Big|_0^t + \int_0^t \int_{\mathbb{T}^d} (\nabla \rho^{\frac{m}{2}})^2 \lesssim \int_0^t \int_{\mathbb{T}^d} g^2.$$
- Caution: Can only be true for non-negative solutions.
- Non-standard weak solutions, rewriting $(*)$ as
$$\partial_t \rho = 2\operatorname{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d$$
- Conclusion: Have to prove uniqueness within this class of solutions.

Ansatz for uniqueness: Show that every weak solution is a renormalized entropy solution (extending the concepts of DiPerna-Lions, Ambrosio to nonlinear PDE).
Let ρ be a weak solution to

$$\partial_t \rho = 2\mathrm{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d.$$

Let

$$\chi(t,x,\xi) = 1_{0<\xi<\rho(x,t)} - 1_{\rho(x,t)<\xi<0}.$$

Then, informally,

$$\partial_t \chi = m\xi^{m-1}\Delta_x \chi - g(x,t)(\partial_\xi \xi^{\frac{m}{2}})\nabla_x \chi + (\nabla_x g(x,t))\xi^{\frac{m}{2}}\partial_\xi \chi + \partial_\xi p$$

with p parabolic defect measure

$$p = \delta(\xi - \rho) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2.$$

- How to make that rigorous? Take convolution
$$\rho^\varepsilon = \varphi^\varepsilon *_x \rho.$$
- Commutator errors,

$$\begin{aligned}
\partial_t \rho^\varepsilon = \varphi^\varepsilon * \partial_t \rho &= \varphi^\varepsilon * (\Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g)) \\
&= \Delta(\varphi^\varepsilon * \rho^m) + \operatorname{div}(\varphi^\varepsilon * (\rho^{\frac{m}{2}} g)) \\
&= \Delta(\rho^\varepsilon)^m + \operatorname{div}((\rho^\varepsilon)^{\frac{m}{2}} g) \\
&+ \Delta(\varphi^\varepsilon * \rho^m) - \Delta(\rho^\varepsilon)^m \\
&+ \operatorname{div}((\varphi^\varepsilon * \rho^{\frac{m}{2}}) g) - \operatorname{div}((\rho^\varepsilon)^{\frac{m}{2}} g) \\
&+ \operatorname{div}(\varphi^\varepsilon * (\rho^{\frac{m}{2}} g)) - \operatorname{div}((\varphi^\varepsilon * \rho^{\frac{m}{2}}) g).
\end{aligned}$$

- Note: Additional commutator errors by commuting convolution and nonlinearities!
- Commutator estimate using non-standard (optimal) regularity $\rho^{\frac{m}{2}} \in L^2_t \dot{H}^1_x$
- Additional renormalization step to compensate low time integrability $\rho^{\frac{m}{2}} g \in L^1_t L^1_x$.

Ansatz for uniqueness: Show that every weak solution is a renormalized entropy solution (extending the concepts of DiPerna-Lions, Ambrosio to nonlinear PDE).

Theorem

A function $\rho \in L_t^\infty L_x^1$ is a weak solution to

$$\partial_t \rho = 2 div(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + div(\rho^{\frac{m}{2}} g)$$

if and only if ρ is a renormalized entropy solution.

Uniqueness for renormalized entropy solutions (variable doubling)

- Additional errors from space-inhomogeneity (with little regularity)
- Note: Entropy dissipation measure

$$q(x,\xi,t) = \delta(\xi - \rho(x,t)) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2$$

  does not satisfy

$$\lim_{|\xi|\to\infty} \int_{t,x} q(x,\xi,t)\, dxdt = 0.$$

- Established arguments [Chen, Perthame; 2003] not applicable.

Theorem (The skeleton equation)

Let $g \in L^2([0,T] \times \mathbb{T}^d; \mathbb{R}^d)$*,* $\rho_0 \in L^1(\mathbb{T}^d)$ *non-negative and* $\int \rho_0 \log(\rho_0) dx < \infty$*,* $m \in [1, \infty)$*.*

1. *There is a unique weak solution*

$$\partial_t \rho = \Delta \rho^m + div(\rho^{\frac{m}{2}} g) \quad on\ \mathbb{R}_+ \times \mathbb{T}^d. \qquad (*)$$

   For two weak solutions $\rho^1, \rho^2 \in L^\infty([0,T]; L^1(\mathbb{T}^1))$ *we have*

$$\|\rho^1 - \rho^2\|_{L^\infty([0,T];L^1(\mathbb{T}^d))} \leq \|\rho_0^1 - \rho_0^2\|_{L^1(\mathbb{T}^d)}.$$

2. *Let* $\{g_n\}_{n\in\mathbb{N}} \subseteq L^2([0,T] \times \mathbb{T}^d; \mathbb{R}^d)$ *with*

$$\lim_{n\to\infty} g_n = g \ \ weakly\ in\ \ L^2([0,T] \times \mathbb{T}^d; \mathbb{R}^d)$$

   and let $\rho_n \in L^1([0,T]; L^1(\mathbb{T}^d))$ *be the corresponding solutions with control* g_n*. Then,*

$$\lim_{n\to\infty} \rho_n = \rho \ \ strongly\ in\ \ L^1([0,T]; L^1(\mathbb{T}^d))$$

   where $\rho \in L^1([0,T]; L^1(\mathbb{T}^d))$ *is the solution with control* g*.*

Consider

$$d\rho^N = \Delta(\rho^N)^m \, dt + \frac{1}{\sqrt{N}} \mathrm{div}\left((\rho^N)^{\frac{m}{2}} \circ dW^{K(N)}(t) \right).$$

Theorem (Large deviation principle)

Let $K(N)$, $n(N) \to \infty$ with $\frac{K(N)^3}{N} \to 0$ for $N \to \infty$. For $\rho_0 \in L^{m+1}(\mathbb{T}^d)$ and $\rho \in L^\infty([0,T]; L^1(\mathbb{T}^d))$ let

$$I_{\rho_0}(\rho) := \inf \left\{ \frac{1}{2} \int_0^T \|g(s)\|^2_{L^2_x} ds : \; g \in L^2_{t,x}, \, \partial_t \rho = \Delta \rho^m + div(\rho^{\frac{m}{2}} g) \right\}.$$

Then, the family $\{\rho^N\}$ satisfies the large deviation principle on $L^\infty([0,T]; L^1(\mathbb{T}^d))$ with good rate function I_{ρ_0}, uniformly on compact subsets of $L^{m+1}(\mathbb{T}^d)$.

K. Dareiotis and B. Gess.
Nonlinear diffusion equations with nonlinear gradient noise.
Electronic Journal of Probability, 25:Paper No. 35, 43, 2020.

N. Dirr, B. Fehrman, and B. Gess.
Conservative stochastic PDE and fluctuations of the symmetric simple exclusion process.
arXiv:2012.02126 [math], Dec. 2020.

B. Fehrman and B. Gess.
Well-posedness of nonlinear diffusion equations with nonlinear, conservative noise.
Archive for Rational Mechanics and Analysis, 233(1):249–322, 2019.

B. Fehrman and B. Gess.
Large deviations for conservative stochastic PDE and non-equilibrium fluctuations.
arXiv:1910.11860 [math], Mar. 2020.

B. Fehrman and B. Gess.
Well-posedness of the Dean-Kawasaki and the nonlinear Dawson-Watanabe equation with correlated noise.
arXiv:2108.08858 [math], Aug. 2021.