1. Introduction

- PREPRINT - SPATIAL ROUGH PATH LIFTS OF STOCHASTIC CONVOLUTIONS

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Abstract. We establish a sufficient condition for finite controlled ρ-variation of the covariance of Gaussian processes with stationary increments, based on concavity of their variance function. This condition allows to lift solutions to a class of fractional stochastic heat equations with additive, possibly colored Wiener noise into geometric Gaussian rough paths with respect to their space variable.

2. Introduction

The lack of spatial regularity of solutions to SPDE often causes serious obstacles concerning well-posedness and stability. As a basic example of such effects one may consider vector-valued stochastic Burgers type equations of the form

$$dX_t^i = \Delta X_t^i + \sum_{j=1}^{n} g_j^i(X_t)\partial_x X_t^j dt + dW_t^i, \quad i = 1, ..., d, \tag{0}$$

where $g_j^i : \mathbb{R}^d \to \mathbb{R}$ are smooth functions and W_t is space-time white noise. Even when $g_j^i \equiv 0$, the solutions to such equations are known to be spatially only β-Hölder continuous for every $\beta < \frac{1}{2}$. Therefore, the term $\partial_x X_t$ is not rigorously defined and a weak formulation has to be used instead. However, as long as g_j^i are not of gradient type one cannot just rely on partial integration in order to pass to a weak formulation. Recently, an alternative approach to (0) based on the theory of rough paths has been developed and successfully applied in order to prove well-posedness and stability (cf. [**?**, 2]). Closely related, instability of spatial discretizations of (0) and the occurrence of correction terms has been observed in [**?**, 3, 4]. The crucial step in the formulation of a weak notion of solution to (0) is the construction of geometric rough paths lifting strictly stationary solutions to the stochastic heat equation in their space variable. I.e., considering

$$d\Psi_t^i = (\Delta - 1)\Psi_t^i dt + dW_t^i, \quad i = 1, ..., d,$$

one needs to construct rough paths $x \mapsto \mathbf{\Psi}(t, x)$ lifting $x \mapsto \Psi(t, x)$. In the special case of stochastic heat equations with space-time white noise on the one dimensional torus, the existence of a corresponding rough path has been shown in [2]. However, the reasoning strongly relied on the simple structure of the equation and on explicit calculations that break down for fractional stochastic heat equations or colored noise. Similar constructions are also fundamental for the recent progress on the KPZ equation [**?**], again on the one dimensional torus.

We provide a general sufficient condition for the existence of a rough path lift of centered, continuous Gaussian processes with stationary increments and concave variance function. As applied to fractional stochastic heat equations, i.e.

$$d\Psi_t^i = (-(-\Delta)^\alpha - 1)\Psi_t^i dt + dW_t^i, \quad \alpha \leq 1, \ i = 1, ..., d,$$

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with possibly colored noise this proves the existence of a geometric rough path, lifting $x \mapsto \Psi(t, x)$ for all $t \geq 0$ under suitable assumptions on the diffusion coefficients.

References

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