Stochastic scalar conservation laws

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joint work with: Panagiotis E. Souganidis
[G., Souganidis; CMS, 2014], [G., Souganidis; arXiv, 2015].

Outline

1. Motivation

2. Well-posedness
   - Spatially homogeneous case
   - Spatially inhomogeneous case

3. Regularity and long-time behavior
   - Long-time behavior
   - Regularization by noise

Motivation

Motivation

Motivation

- We will consider PDE driven by a 'rough' signal z of the type

$$du + \operatorname{div}(A(x,u) \circ dz) = 0.$$

If A is a diagonal matrix this becomes

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(x,u) \circ dz_j = 0$$

- The motivation comes from two directions: Relation to Hamilton-Jacobi equations, mean-field games.

Motivation

- In the one-dimensional case: If v solves the Hamilton-Jacobi equation

$$dv + A(\partial_x v, x) \circ dz = 0$$

then $u = \partial_x v$ solves

$$du + \partial_x A(v, x) \circ dz = 0.$$

- But: The mathematical methods available for Hamilton-Jacobi equations (viscosity solutions) and scalar conservation laws (entropy solutions, kinetic methods) are very different.

Motivation

- Mean-field games going back to Lasry, Lions: Consider the SDE

$$dX_t^i = \sigma\left(X_t^i, \frac{1}{L-1}\sum_{j\neq i}\delta_{X_t^j}\right)\circ dz_t \quad \text{in } \mathbb{R}^N$$

for $i = 1, \ldots, L$.

- Then the empirical law of X converges to a measure π_t with density m_t which evolves according to

$$dm + \operatorname{div}(\sigma^*(x,m)\circ dz) = 0.$$

- Note that in general σ^* is not a diagonal matrix. We need the full generality of

$$du + \operatorname{div}(A(x,u)\circ dz) = 0.$$

Spatially homogeneous case

Well-posedness - Spatially homogeneous case

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Spatially homogeneous case

- We start with the spatially homogeneous case and A being a diagonal matrix, that is:

$$du + \sum_j \partial_{x_j} A(u) \circ dz_j = 0. \tag{SSCL}$$

  Here z is assumed to be a continuous function ('rough' = continuous).
- If z is smooth, then (SSCL) makes sense classically

$$du + \sum_j \partial_{x_j} A_j(u) \dot{z}_j = 0.$$

- Aims:
  - Intrinsic solution: Define solutions to (SSCL) and prove well-posedness.
  - Consistency: Show that solutions to (SSCL) are obtained by approximation of the driving signal z.

Spatially homogeneous case

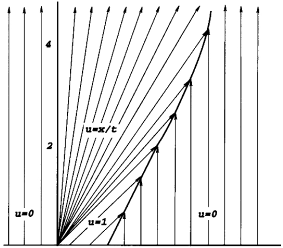
Reminder:

- Solutions to (deterministic) scalar conservation laws

$$du + \sum_j \partial_{x_j} A_j(u)\dot{z}_j = 0$$

  develop shocks (discontinuities)
- e.g. Burgers' equation

$$du + \frac{1}{2}\partial_x u^2 = 0$$
$$u(0) = 1_{[0,1]}$$



- At shocks, weak solutions are non-unique.
- Physically right solution is selected by entropy inequalities [Kružkov, 1970]

$$dS(u) + \sum_j \partial_{x_j} Q_j(u)\dot{z}_j \leq 0.$$

Spatially homogeneous case

- Alternative: kinetic solutions [Lions, Perthame, Tadmor; *JAMS*, 1994].
- For simplicity say $u_0 \geq 0$, which implies $u \geq 0$.
- We consider the characteristic function

$$\chi(t,x,\xi) := 1_{[0,u(t,x)]}(\xi).$$

Elementary calculation (if u were smooth, i.e. no shocks):

$$\begin{aligned}
\partial_t \chi(t,x,\xi) :&= \delta_{\xi=u(t,x)} \partial_t u(t,x) = -\delta_{\xi=u(t,x)} \sum_j \partial_{x_j} A_j(u) \dot{z}_j \\
&= -\delta_{\xi=u(t,x)} \sum_j A_j'(u) \partial_{x_j} u \dot{z}_j = -\delta_{\xi=u(t,x)} \sum_j A_j'(\xi) \partial_{x_j} u \dot{z}_j \\
&= -\sum_j A_j'(\xi) \partial_{x_j} 1_{[0,u(t,x)]}(\xi) \dot{z}_j = -\sum_j A_j'(\xi) \partial_{x_j} \chi(t,x,\xi) \dot{z}_j .
\end{aligned}$$

Spatially homogeneous case

- This is true up to shocks. The shocks introduce an error, the '*entropy dissipation measure*' m:

$$\partial_t \chi(t,x,\xi) + \sum_j A_j'(\xi)\partial_{x_j}\chi(t,x,\xi)\dot{z}_j = \partial_\xi m. \tag{1}$$

- In deterministic setting: u is an entropy solution iff $\chi(t,x,\xi) := 1_{[0,u(t,x)]}(\xi)$ is a kinetic solution to (1).
- Advantage: (1) is a linear equation in χ, at the expense of introducing the additional velocity variable ξ.
- In contrast to the non-linear situation, (1) can be transformed in a 'robust' form, i.e. in a form making sense also for non-smooth z.
- Here we follow the principle idea of stochastic viscosity solutions, i.e. do not transform the PDE itself, but put the transformation into test-functions.

Spatially homogeneous case

- Choose the test-functions φ as solutions to the transport equation

$$\partial_t \varphi(t,x,\xi) + \sum_j A_j'(\xi)\partial_{x_j}\varphi(t,x,\xi)\dot{z}_j = 0. \tag{2}$$

- Then consider *convolutions along characteristics:*

$$\begin{aligned} \partial_t \chi * \varphi &= \partial_t \int \chi(t,x,\xi)\varphi(t,x,\xi)dxd\xi \\ &= \int \left(-\sum_j A_j'(\xi)\partial_{x_j}\chi(t,x,\xi)\dot{z}_j + \partial_\xi m \right) \varphi(t,x,\xi)dxd\xi \qquad (3) \\ &+ \int \chi(t,x,\xi) \left(-\sum_j A_j'(\xi)\partial_{x_j}\varphi(t,x,\xi)\dot{z}_j \right) dxd\xi \\ &= \int \partial_\xi m \varphi(t,x,\xi)dxd\xi. \end{aligned}$$

- The point is that φ in (2) is well-defined also for continuous z, thus (3) is well-defined for z continuous
$\to$ use (3) as the a definition of a solution: *pathwise entropy solution*.

Spatially homogeneous case

- It remains to give meaning to

$$\partial_t \varphi(t,x,\xi) + \sum_j A_j'(\xi)\partial_{x_j}\varphi(t,x,\xi)\dot{z}_j = 0 \tag{4}$$

for continuous signals z.

- Method of characteristics for (4) gives

$$\varphi(t,x,\xi) = \varphi^0(x + A'(\xi)z_t).$$

Theorem (Lions, Perthame, Souganidis; SPDE 2013)

1. *For each $u_0 \in L^\infty \cap BV$ there is a pathwise entropy solution.*
2. *Let $u^{(1)}, u^{(2)} \in L^\infty([0,T]; BV(\mathbb{R}^N))$ be two pathwise entropy solutions with driving signals $z^{(1)}, z^{(2)} \in C_0([0,T];\mathbb{R}^N)$. Then,*

$$\|u^{(1)}(t) - u^{(2)}(t)\|_{L^1(\mathbb{R}^N)} \le \|u_0^1 - u_0^2\|_{L^1(\mathbb{R}^N)} + C\sqrt{\|z^{(1)} - z^{(2)}\|_{C([0,t];\mathbb{R}^N)}}.$$

- In particular, this yields consistency.

Spatially homogeneous case

Comments on the proof:

- Want to estimate

$$\partial_t \int |u^1 - u^2| dy = \partial_t \int |\chi^1 - \chi^2|^2 dy d\xi .$$

- To use definition estimate instead

$$\partial_t \int |\chi^1 * \varphi^\varepsilon - \chi^2 * \varphi^\varepsilon|^2 dy d\xi$$

  with φ^ε test-functions transported along characteristics.

- *Doubling the variables*: One considers a family of testfunctions

$$\varphi^\varepsilon(t,x,y,\xi) = \varphi^{0,\varepsilon}(x - y + A'(\xi)z_t) \xrightarrow[\varepsilon\to 0]{} \delta(x - y + A'(\xi)z_t).$$

- Leads to error terms, due to doubling of the variables, that need to be controled. The simple form of the characteristics allows explicit calculations that lead to crucial cancellations.

Spatially inhomogeneous case

Well-posedness - Spatially inhomogeneous case

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Spatially inhomogeneous case

- Let us now consider the spatially inhomogeneous case:

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(x,u) \circ dz_j = 0.$$

- The principle idea remains the same: We pass to the kinetic formulation:

$$d\chi + \sum_{j=1}^{N} (\partial_u A_j)(x,\xi)\partial_{x_j}\chi \dot{z}_j + \left(\sum_{j=1}^{N} \mathrm{div}_{x_j} A(x,\xi)\dot{z}_j \right) \partial_\xi \chi = \partial_\xi m.$$

- Again we test by solutions to

$$\partial_t \varphi(t,x,\xi) + \sum_{j=1}^{N} (\partial_u A_j)(x,\xi)\partial_{x_j}\varphi \dot{z}_j + \left(\sum_{j=1}^{N} \mathrm{div}_{x_j} A(x,\xi)\dot{z}_j \right) \partial_\xi \varphi = 0.$$

- As before one gets: *convolution along characteristics*

$$\partial_t \chi * \varphi = \int \partial_\xi m \varphi(t,x,\xi) dx d\xi.$$

Spatially inhomogeneous case

- The difficulty lies in the characteristics to

$$\partial_t \varphi(t,x,\xi) + \sum_{j=1}^{N} (\partial_u A_j)(x,\xi)\partial_{x_j}\varphi \dot{z}_j + \left(\sum_{j=1}^{N} \mathrm{div}_{x_j} A(x,\xi)\dot{z}_j \right) \partial_\xi \varphi = 0.$$

- In contrast to the spatially homogeneous case, the characteristics do not have an explicit solution anymore. Instead they are given as the solutions to the controlled DE:

$$dX^i_{(t_1,x,\xi)}(t) = \sum_{j=1}^{M} (\partial_u A_j)(X_{(t_1,x,\xi)}(t), \Xi_{(t_1,x,\xi)}(t))\dot{z}^{t_1,j}(t)dt,$$

$$d\Xi_{(t_1,x,\xi)}(t) = -\sum_{j=1}^{M} (\partial_{x_j} A_j)(X_{(t_1,x,\xi)}(t), \Xi_{(t_1,x,\xi)}(t))\dot{z}^{t_1,j}(t)dt,$$

$$X^i_{(t_1,x,\xi)}(0) = x^i \text{ and } \Xi_{(t_1,x,\xi)}(0) = \xi.$$

- We get

$$\varphi_{t_0}(x,\xi,t) = \varphi^0 \begin{pmatrix} X_{(t,x,\xi)}(t-t_0) \\ \Xi_{(t,x,\xi)}(t-t_0) \end{pmatrix}.$$

Spatially inhomogeneous case

- Hence, to get well-posedness of φ we need stability of (X,Ξ) with respect to the driving signal. I.e. rough path stability.
  $\to$ need z to be a rough path.

Theorem (Gess, Souganidis; CMS, 2015)

1. *For each $u_0 \in L^1 \cap L^2$ there is a pathwise entropy solution.*
2. *Let $u^{(1)}, u^{(2)} \in L^\infty([0,T]; L^1(\mathbb{R}^N))$ be two pathwise entropy solutions with the same driving signal z. Then,*

$$\|u^{(1)}(t) - u^{(2)}(t)\|_{L^1(\mathbb{R}^N)} \leq \|u_0^1 - u_0^2\|_{L^1(\mathbb{R}^N)}.$$

Spatially inhomogeneous case

Comments on the proof:

- One looses the cancellation effect from the homogeneous case.
- Instead, the error has to be carefully controled.
- Key new step: Interval splitting + rough path estimates for the characteristics
- Open question: BV-bounds for the solution. Quantitative continuous dependence on the driving rough path z.

Long-time behavior

Long-time behavior

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Long-time behavior

- We aim to analyze the long-time behavior of

$$du+\sum_{j=1}^{N}\partial_{x_j}A_j(u)\circ d\beta_j=0$$

  on the torus $\mathbb{T}^N$.
- We will show

$$u(t)\to\bar{u}_0=\int_{\mathbb{T}^N}u_0(x)dx\quad\text{for } t\to\infty$$

  in $L^1(\mathbb{T}^N)$.

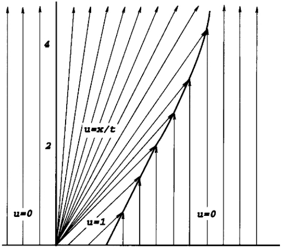
Some results from the deterministic case

- We first recall some (known) results from the deterministic case, i.e. for

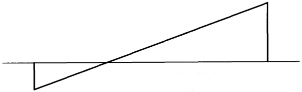
$$\partial_t u + \sum_{j=1}^{N} \partial_{x_j} A_j(u) = 0.$$

- Recall: Burgers' equation

$$du + \frac{1}{2}\partial_x u^2 = 0$$
$$u(0) = 1_{[0,1]}$$



- For large times get asymptotic shape: "N-wave"



Some results from the deterministic case

Some existing results:

- $N = 1$: [Lax; *CPAM*, 1957]. Rate for strictly convex flux:

$$\|u(t) - \bar{u}_0\|_\infty \lesssim t^{-1}.$$

- $N = 2$: [Engquist, E; *CPAM*; 1993]. Rate for strictly convex flux:

$$\|u(t) - \bar{u}_0\|_\infty \lesssim t^{-1}.$$

- $N \geq 1$: [Chen, Frid; *ARMA*; 1999], [Chen, Perthame; *Proc. AMS*; 2009]. If A is 'genuinely nonlinear' then

$$u(t) \to \bar{u}_0 := \int_{\mathbb{T}^N} u_0 \, dx \quad \text{for } t \to \infty,$$

in $L^1(\mathbb{T}^N)$.

- For general $N \geq 3$ no rate of convergence known!

Some results from the deterministic case

Idea of the proof for general $N \geq 1$:

- Consider pullback limit, i.e. let $u(s,t,u_0)$ be the solution started in u_0 at time $s \leq 0$.
- Key point: By averaging Lemma (genuine nonlinearity of A), the solution operator $S_t : L^\infty \to L^1$ is locally compact. Thus one can extract a subsequence such that

$$u(s,t,u_0) \to v(t)$$

  for $s \to -\infty$.
- The limit v is a solution to

$$\partial_t v + \sum_{j=1}^{N} \partial_{x_j} A_j(v) = 0$$

  for all time $t \in \mathbb{R}$.
- But such a function has to be constant (again via averaging techniques).

(New) rates for the deterministic case

- Assume that the flux A is genuinely nonlinear, in the sense that: there exist $\theta \in (0,1]$ and $C > 0$ such that, for all $\sigma \in S^{N-1}$, $z \in \mathbb{R}$ and $\varepsilon > 0$,

$$|\{\xi \in \mathbb{R} : |A'(\xi) \cdot \sigma - z| \leq \varepsilon\}| \leq C\varepsilon^{\theta}.$$

- For example: For A strictly convex, $N = 1$ we have $\theta = 1$.
- Let u be the unique entropy solution to

$$\partial_t u + \sum_{j=1}^{N} \partial_{x_j} A_j(u) = 0.$$

Theorem (G., Souganidis; 2015)

For $t \geq 1$ and $u_0 \in L^2(\mathbb{T}^N)$,

$$\|u(\cdot, t; \cdot, u_0) - \bar{u}_0\|_1 \leq t^{-\frac{\theta}{2+\theta}} \left(\|u_0\|_2^2 + 1\right).$$

(New) rates for the stochastic case

- Let us return to

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(u) \circ d\beta_j = 0.$$

- Again assume that A is genuinely nonlinear.

Theorem (G., Souganidis; 2015)

For $t \geq 1$ and $u_0 \in L^2(\mathbb{T}^N)$,

$$\mathbb{E}\|u(\cdot,t;\cdot,u_0) - \bar{u}_0\|_1 \leq \left(\|u_0\|_2^2 + 1\right) t^{-\frac{\theta}{3+\theta}}.$$

- E.g. $\theta = 1$: deterministic rate $t^{-\frac{1}{3}}$, stochastic rate $t^{-\frac{1}{4}}$. But: No claim of optimality.
- Note: Brownian motion scales like $\sqrt{t}$, which "slows down" characteristics.

Regularization by noise

Regularization by noise

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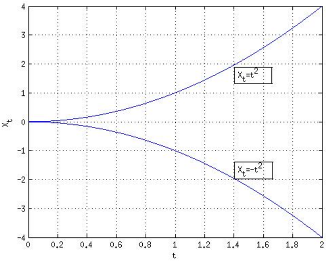
Regularization by noise - linear case

- We recall: Consider
$$du + b(x) \cdot \nabla u = 0 \qquad \text{(TE)}$$
for non-Lipschitz b (but, say, Hölder continuous). E.g. $b(x) = \text{sgn}(x)\sqrt{|x|}$.
- In general, characteristics collide causing shocks (i.e. discontinuities). Solution is not better than $u(t) \in BV$ even if u_0 is smooth.
- Weak solutions are non-unique: e.g. $b(x) = \text{sgn}(x)\sqrt{|x|}$



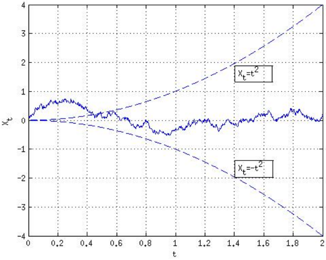
Regularization by noise - linear case

- In contrast: Consider

$$du + b(x) \cdot \nabla u = -\nabla u \circ dW_t. \qquad \text{(STE)}$$

- Two (related) effects: regularization by noise, well-posedness by noise.
- Regularization by noise [Flandoli, Fedrizzi; *JFA*, 2013]: If u_0 is smooth then $u(t)$ is smooth.
- Well-posedness by noise [Flandoli, Gubinelli, Priola; *Invent. Math.*, 2010]: Weak solutions to (STE) are unique.

$$du + b(x) \cdot \nabla u = -\nabla u \circ dW_t$$
$$b(x) = \text{sgn}(x)\sqrt{|x|}$$



- Entirely open: What about the nonlinear case, e.g. Burgers?

Quasi-solutions and averaging

- Consider the Burgers' equation

$$\partial_t u + \partial_x u^2 = 0 \quad \text{on } \mathbb{T} \tag{B}$$

- Weak solutions to (B) are not unique.
- We consider quasi-solutions [De Lellis, Otto, Westdickenberg; *ARMA*, 2003]: A weak solution u to (B) is a quasi-solution, if for some Radon measure m

$$\partial_t \chi + \xi \partial_x \chi = \partial_\xi m.$$

  Quasi-solutions to (B) are not unique.
- Recall: entropy solutions are kinetic solutions, i.e. for some non-negative measure m

$$\partial_t \chi + \xi \partial_x \chi = \partial_\xi m.$$

Quasi-solutions and averaging

- By averaging techniques [Jabin, Perthame; 2002]: Let u be a quasi-solution to (B). Then
$$u(t) \in W^{1,\lambda} \quad \text{for all } \lambda \in (0, \frac{1}{3}).$$
- [De Lellis, Westdickenberg; *AIHP*, 2003]: There is a quasi-solution to (B) such that
$$u(t) \notin W^{1,\lambda} \quad \text{for all } \lambda > \frac{1}{3},$$
i.e. regularity by averaging is sharp for quasi-solutions.
- Question: Does noise improve the situation?

(New) results for the stochastic case

- Consider the stochastic Burgers' equation

$$du + \partial_x u^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T} \tag{SB}$$

Theorem (G., Souganidis; 2015)

Let u be a pathwise quasi-solution to (SB). Then, $t > 0$,

$$u(t) \in W^{\lambda,1} \quad \text{for all } \lambda \in (0, \frac{1}{2}), \mathbb{P}\text{-a.s.}.$$

- Thus: quasi-solutions to (SB) are more regular than to (B), i.e. regularization by noise.

Thanks

Thanks!