Well-posedness by noise for scalar conservation laws

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[G., Maurelli.; CPDE 2019+], [Gassiat, G., Lions, Souganidis.; arxiv 2018], [Smith, G..; JMPA 2019+], [Gassiat, G..; PTRF 2018].

Outline

- Classical well-posedness for ODE:

$$\begin{aligned} dX_t^x &= b(X_t^x)dt \\ X_0^x &= x \end{aligned}$$

is well-posed if b is sufficiently smooth, e.g. Lipschitz continuous.

- In contrast, well-posedness for SDE: $(\sigma > 0)$

$$\begin{aligned} dX_t^x &= b(X_t^x)dt + \sigma d\beta_t \\ X_0^x &= x \end{aligned}$$

has a unique solution if b is bounded, measurable. This is called '*well-posedness by noise*'.

- Consider
$$du + b(x)\cdot\nabla u\,dt = \sigma\nabla u\circ d\beta_t. \tag{STE}$$
- Characteristics for (STE):
$$\begin{aligned} dX_t^x &= b(X_t^x)dt - \sigma d\beta_t \in \mathbb{R}^d \\ X_0^x &= x. \end{aligned}$$
- Two (related) effects: regularization by noise, well-posedness by noise.
- Well-posedness by noise [Flandoli, Gubinelli, Priola; 2010]: Weak solutions to (STE) are unique.
- Regularization by noise [Flandoli, Fedrizzi; 2013]: If u_0 is smooth then $u(t)$ is smooth.

- Left open: What about the nonlinear case, e.g. Burgers?
- Linear multiplicative noise does not help anymore:

$$du + \partial_x u^2 \, dt = \partial_x u \circ d\beta_t.$$

Then: $v(t,x) := u(t, x - \beta_t)$ is a solution to

$$\partial_t v + \partial_x v^2 = 0.$$

- In particular, weak solutions are non-unique.
- Conclusion in [Flandoli, Gubinelli, Priola; *Invent. Math.*, 2010]:
  „It is very easy to produce examples [...] for a stochastic version of Euler equation which show that the particular noise we use does not have any regularizing effect in this case. [...] The generalization to nonlinear transport equations, where b depends on u itself, would be a major next step for applications to fluid dynamics but it turns out to be a difficult problem.“

Well-posedness by noise for stochastic scalar conservation laws

- Consider
$$\partial_t u + b(x,u)\cdot\nabla u = 0.$$
- Scalar conservation laws with irregular flux: Traffic flows, sedimentation processes
[De Philippis et al., *CPDE*, 2015; Andreianov, Karlsen, Risebro, *ARMA*, 2011].
- In this talk: For simplicity consider
$$\partial_t u + b(x)\cdot\nabla(u^2) = 0,$$
for irregular b (in particular $\mathrm{div} b \notin L^\infty$).
- The deterministic problem is ill-posed in general: Entropy solutions are non-unique.

- Model example: Consider

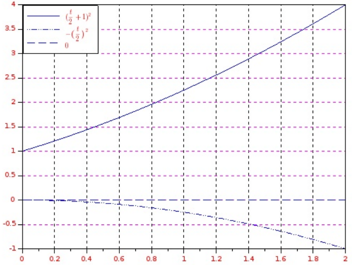
$$\partial_t u + b(x)\cdot\nabla(u^2) = 0 \qquad (\star)$$
$$u(0,x) = 1_{[0,1]}(x)$$

with $b(x) = 2\mathrm{sgn}(x)(\sqrt{|x|}\wedge K)$.

- Rankine-Hugoniot implies:

$$u^1(t,x) := \begin{cases} 1 & \text{für } 0 \le x \le \left(\frac{t}{2}+1\right)^2 \\ 0 & \text{sonst,} \end{cases}$$

$$u^2(t,x) := \begin{cases} 1 & \text{für } -\left(\frac{t}{2}\right)^2 \le x \le \left(\frac{t}{2}+1\right)^2 \\ 0 & \text{sonst.} \end{cases}$$



- Can we restore well-posedness by adding a linear multiplicative noise term?
- Non-trivial: shocks due to the nonlinearity and shocks due to the irregularity of b may combine in such a way that this noise may be insufficient.
- Stochastic Burgers' equation:

$$du + b(x) \cdot \nabla(u^2)dt = \nabla u \circ d\beta_t.$$

Theorem

Assume $b \in (W^{1,1} \cap L^\infty)(\mathbb{R}^d)$ and $\operatorname{div} b \in L^p(\mathbb{R}^d)$ for some $p > d$. For $u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$, the stochastic Burgers' equation admits a unique entropy solution.

Model example: $b(x) = 2\operatorname{sgn}(x)(\sqrt{|x|} \wedge K)$.

- Kinetic solutions: Given $u = u(t,\omega,x)$, introduce a new (velocity) variable $\xi \in \mathbb{R}$ and define the kinetic function:

$$f = f[u](t,\omega,x,\xi) = 1_{\xi < u(t,x)} - 1_{\xi < 0}.$$

- Then u is an entropy solution iff

$$\partial_t f + 2b(x)\xi \cdot \nabla_x f - \nabla_x f \circ d\beta_t = \partial_\xi m.$$

Definition

$f = f(t,\omega,x,\xi)$ is a generalized entropy solution if f solves the kinetic equation

$$\partial_t f + 2b(x)\xi \cdot \nabla_x f - \nabla_x f \circ d\beta_t = \partial_\xi m$$

and, for some nonnegative measure ν,

$$|f| = \mathrm{sgn}(\xi) f \le 1 \quad \partial_\xi f = \delta_0 - \nu.$$

- Set of generalized entropy solutions is convex and closed under weak limits.

Proposition

Let $u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$ and b, $divb \in L^1(\mathbb{R}^d)$. Then there exists a generalized entropy solution.

- Proof of uniqueness:
  - First step (deterministic): Via renormalization arguments derive an inequality for $|f| - f^2$ (similar to the kinetic equation).
  - Second step (stochastic): Take the expectation and use *parabolic* theory.
- Kinetic equation

$$\partial_t f + 2b(x)\xi \cdot \nabla_x f - \nabla_x f \circ d\beta_t = \partial_\xi m$$

rewritten in Itô form:

$$\partial_t f + 2b(x)\xi \cdot \nabla_x f - \nabla_x f \, d\beta_t - \frac{1}{2}\Delta_x f = \partial_\xi m.$$

A Laplacian appears, which suggests regularization. Note: the equation is hyperbolic (not parabolic: no regularization of initial datum).

- Taking the expectation yields

$$\partial_t \mathbb{E} f + 2b(x)\xi \cdot \nabla_x \mathbb{E} f - \frac{1}{2}\Delta_x \mathbb{E} f = \partial_\xi \mathbb{E} m,$$

i.e. a parabolic PDE.

Proposition

Fix $T > 0$ *and assume* $b \in L^\infty(\mathbb{R}^d)$, $divb$ *in* $L^p(\mathbb{R}^d)$ *for some* $p > d$. *Then, there exists a* $\varphi \geq 0$, *independent of* ξ, *with* $\varphi_T \sim 1$, *such that*

$$\partial_t \varphi - 2div_x(b(x)\xi\varphi) + \frac{1}{2}\Delta_x \varphi \leq C$$

for some $C > 0$ *(independent of* T*).*

Theorem

Assume $b \in (W^{1,1} \cap L^\infty)(\mathbb{R}^d)$ *with* $divb \in L^p$ *for some* $p > d$. *Then*

1. *Every generalized entropy solution is an entropy solution.*
2. *For* $u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$ *there is an entropy solution.*

Entropy solutions are unique

Well-posedness by nonlinear noise

- Consider
$$\partial_t u + \nabla \cdot (b(t,x)u) + \nabla B(u) \circ d\beta_t = 0$$
on $[0,T] \times \mathbb{R}^d$.
- The noise $\nabla B(u) \circ d\beta_t$ may degenerate: $b(u) := B'(u)$ may vanish for some values of u.

We assume

- [Renormalization] $b \in L^1_t(W^{1,1}_x)$ and $\mathrm{div} b \in L^r_{t,x}$ for some $r > 1$.
- [Sub-criticality] $b \in L^\infty_{t,x}$ and $(\mathrm{div} b)_- \in L^q_{t,x}$ for some $q > d+2$. Moreover, there exists an $R > 0$ such that $\mathrm{supp} b \subseteq [0,T] \times B_R$.
- [Asymptotic ellipticity] $B \in C^3(\mathbb{R};\mathbb{R})$. For $b(\rho) := B'(\rho)$, there exist strictly positive λ and Λ such that
$$\sup_{\rho \in \mathbb{R}} b^2(\rho) \leq \Lambda \quad \liminf_{|\rho| \to \infty} b^2(\rho) = \lambda.$$

Theorem

Under the above assumptions: For $u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$ there exists a unique kinetic solution to starting from u_0.

Regularization by noise for stochastic Hamilton-Jacobi equations

Regularization by noise for stochastic Hamilton-Jacobi equations

- Can we use nonlinear noise to regularize nonlinear PDE?
- Model example: Porous medium equation

$$\partial_t w = \frac{1}{6}\partial_{xx} w^3, \quad \text{on } \mathbb{R}$$

with smooth, compactly supported initial condition.

- Solutions are known to develop singularities:

$$\|\partial_x w(t)\|_{L^\infty} = \infty$$

for all $t > 0$ large enough.

- Linear multiplicative noise does not help:

$$dv = \frac{1}{6}\partial_{xx} v^3\, dt + \sigma \partial_x v \circ d\beta_t, \quad \text{on } \mathbb{R}.$$

Then $w(t,x) = v(t, x - \sigma\beta_t)$.

- Instead, consider, for $\sigma > 0$,

$$dv = \frac{1}{6}\partial_{xx} v^3\, dt + \sigma \frac{1}{2}\partial_x v^2 \circ d\beta_t, \quad \text{on } \mathbb{R}. \tag{SPME}$$

- Note: If u is the viscosity solution to

$$du = \frac{1}{6}\partial_x(\partial_x u)^3\, dt + \sigma \frac{1}{2}(\partial_x u)^2 \circ d\beta_t, \quad \text{on } \mathbb{R},$$

then, $v = \partial_x u$ solves (SPME).

- General framework: Consider

$$du = F(t,x,u,Du,D^2u)\,dt + \frac{1}{2}|Du|^2 \circ d\beta_t, \quad \text{on } \mathbb{R}^N,$$

  and compare to the regularity of non-perturbed problem

$$dw = F(t,x,w,Dw,D^2w), \quad \text{on } \mathbb{R}^N.$$

- F satisfies the usual assumptions from the theory of stochastic viscosity solutions

- Control on the rate of loss of semiconcavity: There is a $V_F \in Lip_{loc}(\mathbb{R}_+ \setminus \{0\})$ such that, for $\ell_0 > 0$,

$$D^2 g \le \frac{Id}{\ell_0} \;\Rightarrow\; D^2(S_F(t,g)) \le \frac{Id}{\ell(t)},$$

where $t \mapsto S_F(t,g)$ denotes the solution to

$$\begin{aligned} \partial_t w + F(t,x,w,Dw,D^2 w) &= 0 \\ w(0) &= g \end{aligned}$$

and ℓ the solution to

$$\begin{aligned} \dot{\ell}(t) &= V_F(\ell(t)) \\ \ell(0) &= \ell_0. \end{aligned}$$

Theorem

Let u be the solution to

$$\partial_t u = F(t,x,u,Du,D^2 u) + \frac{1}{2}|Du|^2 \circ d\beta_t,$$

with

$$D^2 u_0 \leq \frac{Id}{\ell_0} \quad \text{some } \ell_0 \in [0,\infty).$$

Then for each $t \geq 0$, one has

$$D^2 u(t,\cdot) \leq \frac{Id}{L_t},$$

where L is the maximal solution to

$$dL_t = V_F(L_t)dt + d\beta_t, \;\; \text{with } L_t \geq 0,$$
$$L_0 = \ell_0.$$

- Return to the model example

$$dv = \frac{1}{6}\partial_{xx} v^3\, dt + \sigma \frac{1}{2}\partial_x v^2 \circ d\beta_t, \quad \text{on } \mathbb{R}.$$

- Determinstic case: $\|\partial_x w(t)\|_\infty = \infty$ for all $t > 0$ large enough.
- We have the sharp bound

$$\|\partial_x v(t)\|_\infty \le \frac{1}{L^+(t) \wedge L^-(t)},$$

where $L^\pm$ solve

$$dL^\pm = -\frac{1}{L^\pm(t)}\, dt \pm \sigma d\beta_t, \text{ with } L^\pm_t \ge 0,$$
$$L^\pm(0) = \frac{1}{\|(\partial_x v_0)_\pm\|_\infty}.$$

- In conclusion,
  - If $\sigma^2 > 2$: For all $t \geq 0$, $\mathbb{P}$-a.s.

$$v(t) \in W^{1,\infty}$$

  - If $\sigma^2 \leq 2$: $\mathbb{P}$-a.s. for all $t > 0$ large enough

$$v(t) \notin W^{1,\infty}$$

- Observation: As long as the solution is Lipschitz continuous, it is time reversible.

Speed of propagation for stochastic Hamilton-Jacobi equations

Speed of propagation for stochastic Hamilton-Jacobi equations

- Consider

$$du = H(Du, x)\cdot d\xi \ \text{ in } \ \mathbb{R}^d \times (0,T] \qquad u(\cdot,0) = u_0 \ \text{ in } \ \mathbb{R}^d,$$

with $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ convex and Lipschitz continuous in the first argument, $\xi \in C_0([0,T])$.
- By Lions-Souganidis there exists a unique viscosity solution.
- Given $T > 0$ let

$$\rho_H(\xi, T) := \sup \Big\{ R \geq 0 : \text{ there exist solutions } u^1, u^2 \text{ and } x \in \mathbb{R}^d, \\ \text{such that } u^1(\cdot,0) = u^2(\cdot,0) \text{ in } B_R(x) \text{ and } u^1(x,T) \neq u^2(x,T) \Big\},$$

where $B_R(x)$ is the ball in $\mathbb{R}^d$ centered at x with radius R.
- Classical (e.g. Crandall, Lions): If ξ is a BV-path, then

$$\rho_H(\xi, T) \leq L \|\xi\|_{TV([0,T])},$$

where $\|\xi\|_{TV([0,T])}$ is the total variation semi-norm of ξ and L is the Lipschitz constant of H. It is easy to see that this is sharp when $\dot{\xi} \equiv 1$.

- General continuous signal ξ: If $H(p,x) = H_1(p) - H_2(p)$, where H_1, H_2 convex, Lipschitz with Lipschitz constant L and $H_1(0) = H_2(0) = 0$, then, for any constant A, if

$$u(0,\cdot) \equiv A \text{ on } B_R(0),$$

then

$$u(t,\cdot) \equiv A \text{ on } B_{R(t)}(0), \quad \text{for } R(t) := R - L(\max_{s\in[0,t]} \xi(s) - \min_{s\in[0,t]} \xi(s)).$$

- This does not imply a finite domain of dependence.
- In fact: For $H(p) = |p_1| - |p_2|$ equality is attained in

$$\rho_H(\xi, T) \leq L\|\xi\|_{TV([0,T])},$$

for all continuous ξ.

- No finite domain of dependence if $\xi \notin BV([0,T])$

Given $\xi \in C_0([0,T])$, the sequence $(\tau_i)_{i\in\mathbb{Z}}$ of successive extrema of ξ:

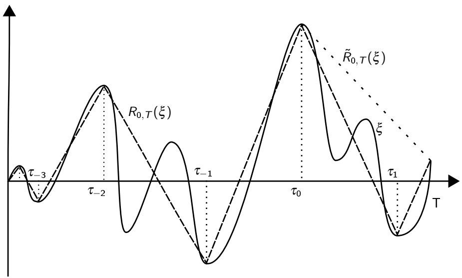


Figure: The (fully) reduced path

Definition

(i) The reduced path $R_{0,T}(\xi)$ is a piecewise linear function which agrees with ξ on $(\tau_i)_{i\in\mathbb{Z}}$.
(ii) The fully reduced path $\tilde{R}_{0,T}(\xi)$ is a piecewise linear function agreeing with ξ on $(\tau_{-i})_{i\in\mathbb{N}} \cup \{T\}$.

Recall:

$$du = H(Du,x)\cdot d\xi \quad \text{in} \quad \mathbb{R}^d \times (0,T] \qquad u(\cdot,0) = u_0 \quad \text{in} \quad \mathbb{R}^d.$$

Let $S^\xi(s,t)$ be the nonlinear semigroup of solutions.

Theorem

Let $\xi \in C_0([0,T])$. Then

$$S^\xi(0,T) = S^{R_{0,T}(\xi)}(0,T).$$

Hence

$$\rho_H(\xi,T) \le L\, \|R_{0,T}(\xi)\|_{TV([0,T])}.$$

Theorem

Let B be a Brownian motion, $T > 0$. Then, a.s.

$$\left\|R_{0,T}(B)\right\|_{TV([0,T])} < \infty.$$

Sharpness of the upper bound:

Theorem

Let $H(p) = |p|$ on $\mathbb{R}^d$ with $d \geq 1$. Then, for all $T > 0$ and $\xi \in C_0([0,T])$,

$$\rho_H(\xi, T) \geq \|\tilde{R}_{0,T}(\xi)\|_{TV([0,T])}.$$

When $d = 1$, then

$$\rho_H(\xi, T) = \|\tilde{R}_{0,T}(\xi)\|_{TV([0,T])}.$$

Proof.

By keeping track of explicit cancellations. □

Sharp speed of propagation in higher dimension?

Consider $H(p) = |p|$ and

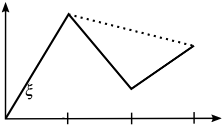
$$\dot{\xi}(t) = \begin{cases} 4 & \text{for } (0,1) \\ -2 & \text{on } (1,2) \\ 1 & \text{on } (2,3) \end{cases} \quad \text{and } \dot{\tilde{R}}_{0,3}(\xi) = \begin{cases} 4 & \text{on } (0,1) \\ -\frac{1}{2} & \text{on } (1,3); \end{cases}$$

Which gives

Figure: Evolution of $S^{\xi}_{|\cdot|}(0,t)$ at $t = 0,\ t = 1,\ t = 2,\ t = 3$

Figure: Evolution of $S^{\tilde{R}(\xi)}_{|\cdot|}(0,t)$ at $t = 0,\ t = 1,\ t = 3$

Let $\delta_1 > \delta_2 > \delta_3 > 0$ and ξ continuous on $[0,3]$ with



$$\dot{\xi} = \begin{cases} \delta_1 & \text{on } (0,1), \\ -\delta_2 & \text{on } (1,2), \\ +\delta_3 & \text{on } (2,3). \end{cases}$$

Then

$$\rho_{|\cdot|}(\xi,3) = \delta_1 + \delta_2 + \delta_3 = \|\xi\|_{TV([0,3])} = \|R_{0,T}(\xi)\|_{TV([0,T])} > \|\tilde{R}_{0,T}(\xi)\|_{TV([0,T])}.$$

Figure: Evolution of $S^{\xi}_{|.|}(0,\cdot)P_1$

Figure: Evolution of $S^{\xi}_{|\cdot|}(0,\cdot)P_2$