Optimal regularity for the porous medium equation

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[G.; JEMS 2019+], [G., Sauer, Tadmor; arxiv, 2019].

Outline

- We consider the porous medium equation

$$\partial_t u = \Delta \left(|u|^{m-1}u\right) \text{ on } (0,T)\times\mathbb{R}^d$$
$$u(0) = u_0 \text{ on } \mathbb{R}^d,$$

  with $u_0 \in L^1(\mathbb{R}^d)$, $m>1$.

- Degenerate parabolic Anderson model

$$\partial_t u = \Delta \left(|u|^{m-1}u\right) + u\xi \quad \text{on } (0,T)\times\mathbb{R}$$

  with $u_0 \in L^1(\mathbb{R})$, $m\in(1,2)$, ξ spatial white noise.

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  with $u_0 \in L^1(\mathbb{R})$, $m \in (1,2)$, ξ spatial white noise.
- Aim: Optimal regularity of solutions in (fractional) Sobolev spaces.

Application: Population dynamics

- Spreading of biological populations

$$\partial_t u = \mathrm{div}(\kappa \nabla u) + f(u),$$

  where u is the density of the species, $f(u)$ is the reproduction/death rate.

- If populations avoid crowding κ is an increasing function of the population density, $\kappa = \varphi(u)$ with φ increasing.
- In particular cases we have $\varphi(u) = au^{\gamma}$. Hence,

$$\partial_t u = \frac{a}{\gamma+1} \Delta u^{\gamma+1} + f(u),$$

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$$\partial_t u = \frac{a}{\gamma+1}\Delta u^{\gamma+1} + f(u),$$

- Random environment leads to the degenerate parabolic Anderson model

$$\partial_t u = \frac{a}{\gamma+1}\Delta u^{\gamma+1} + u\xi,$$

where ξ is spatial white noise.

Application: Interacting particles

- Interacting particle system

$$\frac{d}{dt}X_t^i = -\frac{1}{L}\sum_{j=1, j\neq i}^{L} \nabla V_L(X_t^i - X_t^j) \quad i = 1 \dots L,$$

  where V_L is a rescaled interaction potential (repelling)

$$V_L(x) = \lambda^d V_1(\lambda x), \lambda = L^{\frac{\beta}{d}}$$

  and $\beta \in (0,1)$.

- Consider the empirical process

$$t \mapsto \mu_t^L = \frac{1}{L}\sum_{i=1}^{L} \delta_{X_t^i}.$$

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Under regularity, decay and symmetry assumptions on V_1 obtain

Theorem (Oelschläger)

If $\mu_0^L \rightharpoonup m_0(x)dx$*, then* $\mu_t^L \rightharpoonup m_t(x)dx$ *and with* $c = \frac{1}{2}\int V_1(x)\,dx$*,*

$$\partial_t m = c\Delta m^2, \quad m(0) = m_0.$$

See also: Lions-Mas Gallic 2001, Figalli-Philipowski 2008, Carrillo-Craig-Papacchini 2018

Scaling arguments and special solutions

Scaling arguments and special solutions

- Note

$$\partial_t u = \Delta u^{[m]} = m\mathrm{div}(|u|^{m-1}\nabla u)$$
$$= m|u|^{m-1}\Delta u + m(m-1)u^{[m-2]}|\nabla u|^2.$$

- Barenblatt solution:

$$U(x,t) = t^{-\alpha}F(xt^{-\beta}) = t^{-\alpha}(C - k|xt^{-\alpha/d}|^2)_+^{\frac{1}{m-1}},$$

where $\alpha = \frac{d}{d(m-1)+2}$, $k = \frac{(m-1)\alpha}{2md}$. We observe that

$$lim_{t\downarrow 0} U(x,t) = M\delta_0(x)$$

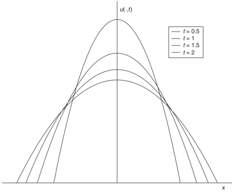

Figure: Fundamental solution of the porous medium equation

Lemma

Assume that for some $s \geq 0$, $p \geq 1$, $C \geq 0$ we have

$$\|u\|^p_{L^p([0,T];\dot{W}^{s,p}(\mathbb{R}^d_x))} \leq C\|u_0\|_{L^1(\mathbb{R}^d_x)},$$

for all solutions u to PME. Then, necessarily $p \leq m$ and $s \leq \frac{2}{m}$.

Use scale invariances:

$$\tilde{u}(t,x) := u(\eta t,x)\eta^{\frac{1}{m-1}}, \quad \tilde{u}(t,x) := u(t,\eta x)\eta^{-\frac{2}{m-1}}.$$

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Example

Consider the Barenblatt solution

$$U(t,x) = t^{-\alpha}(C - k|xt^{-\beta}|^2)_+^{\frac{1}{m-1}}.$$

Then

$$U \in L^m([0,T];\dot{W}^{s,m}(\mathbb{R}^d_x))$$

implies $s < \frac{2}{m}$.

Use $U(t,x) = t^{-\alpha}F(xt^{-\beta})$.

Existing regularity results

Existing regularity results

- Continuity
  Caffarelli-Friedman 1979, Sacks 1983, Cafarelli-Evans 1983, DiBenedetto 1983, Ziemer 1982
- Hölder continuity: α-Hölder continuity with $\alpha = \frac{1}{m} \in (0,1)$.
  Caffarelli-Friedman 1980, DiBenedetto-Friedman 1985, Bögelein, Duzaar, Gianazza 2014
- Regularity of the open interface
  Caffarelli-Friedman 1980, Caffarelli-Vazquez-Wolansky 1987, Caffarelli-Wolanski 1990, Daskalopoulos-Hamilton 1998, Koch 1999
- Eventual C^∞ regularity
  Aronson-Vázquez 1987, Kienzler-Koch-Vazquez 2016
- Regularity of the pressure or powers of the solution
  Koch 1999, Gianazza-Schwarzacher 2016
- Time regularity (vanishing force)
  Aronson-Bénilan 1979, Crandall-Pazy-Tartar 1979, Bénilan-Crandall 1981, Crandall-Pierre 1982
- Regularity in Sobolev spaces
  Lions-Perthame-Tadmor 1994, Ebmeyer 2005, Tadmor-Tao 2007

Let $\dot{\mathcal{N}}^{s,p}$ be the homogeneous Nikolskii space ($\dot{\mathcal{N}}^{s,p} = \dot{B}^s_{p,\infty}$).

Theorem (Tadmor, Tao; CPAM 2007, Ebmeyer; JMAA 2005)

Let $u_0 \in L^2(\mathbb{R}^d_x)$. Then

$$\|u\|^{m+1}_{L^{m+1}([0,T];\dot{\mathcal{N}}^{\frac{2}{m+1},m+1}(\mathbb{R}^d_x))} \le C_m \|u_0\|^2_{L^2_x}.$$

- Note: $\frac{2}{m+1} \le 1$, which is inconsistent with the linear case $(m=1)$ and with the optimal regularity of the Barenblatt solution.

Consider

$$\partial_t u = \Delta u^{[m]} + S(t,x).$$

By (soft) energy methods may be improved to:

Theorem (G. 2019+)

Let $\varepsilon > 0$, $m \geq 2$ and $u_0 \in L^{1+\varepsilon}(\mathbb{R}^d_x)$, $S \in L^{1+\varepsilon}([0,T] \times \mathbb{R}^d_x)$. Then

$$\|u\|^{m+\varepsilon}_{L^{m+\varepsilon}([0,T];\dot{\mathcal{N}}^{\frac{2}{m+\varepsilon},m+\varepsilon}(\mathbb{R}^d_x))} \leq C_{\varepsilon,m}\|u_0\|^{1+\varepsilon}_{L^{1+\varepsilon}_x}.$$

- Note: optimal regularity for the Barenblatt solution, but $m \geq 2$ implies $\frac{2}{m+\varepsilon} < 1$.
- Problem: How to get to more than one derivative?

Optimal regularity for the porous medium equation

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Consider

$$\partial_t u = \frac{1}{m}\Delta u^{[m]} + S(t,x) \quad \text{on } (0,T) \times \mathbb{R}^d_x \tag{PME}$$

with $u_0 \in L^1(\mathbb{R}^d_x)$, $S \in L^1([0,T] \times \mathbb{R}^d_x)$.

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Theorem (G., 2017)

Let $\varepsilon > 0$, $u_0 \in L^{1+\varepsilon}(\mathbb{R}^d_x)$, $S \in L^{1+\varepsilon}([0,T]\times\mathbb{R}^d_x)$. Let u be the unique entropy solution to the PME. Then, for all

$$s \in [0, \frac{2}{m}), \quad p \in [1, m)$$

we have

$$u \in L^p([0,T]; \dot{W}^{s,p}_{loc}(\mathbb{R}^d_x)).$$

In addition, for all $\mathcal{O} \subset\subset \mathbb{R}^d$ there is a constant $C = C(m,p,s,T,\mathcal{O})$ such that

$$\|u\|_{L^p([0,T];\dot{W}^{s,p}(\mathcal{O}))} \le C\left(\|u_0\|^2_{L^1_x} + 1\right).$$

"Proof": A real analysis attempt

- Kinetic form [Lions, Perthame, Tadmor 1994], [Chen, Perthame; 2003]: Introduce
$$\chi(u(t,x),v) = 1_{v<u(t,x)} - 1_{v<0}.$$
Then,
$$\partial_t \chi = |v|^{m-1}\Delta_x \chi + \partial_v q \text{ on } (0,T)\times\mathbb{R}^d_x\times\mathbb{R}_v,$$
for some $q \in \mathscr{M}^+$.
- Variation of constants/Duhamel
$$\chi(t,x,v) = e^{-|v|^{m-1}t\Delta}\chi_0(x,v) + \int_0^t e^{-|v|^{m-1}(t-r)\Delta}\partial_v q(r,x,v)dr.$$
- Decompose u in degenerate and non-degenerate part:
$$u(t,x) = \int_v \chi(u(t,x),v) = \underbrace{\int_{|v|\le\lambda}\chi(u(t,x),v)}_{u^1(t,x)} + \underbrace{\int_{|v|\ge\lambda}\chi(u(t,x),v)}_{u^2(t,x)}.$$
- Note:
$$u^2(t,x) = \int_{|v|\ge\lambda} e^{-|v|^{m-1}t\Delta}\chi_0(x,v) + \int_0^t\int_{|v|\ge\lambda} e^{-|v|^{m-1}\Delta(t-r)}\partial_v q(r,x,v).$$

- Trivial estimate: For all $r \geq 1$,

$$\|u^1\|_{L^r_{t,x}} = \left\| \int_{|v|\leq\lambda} \chi(u(t,x),v) \right\|_{L^r_{t,x}} \lesssim \lambda.$$

- Recall:

$$u^2(t,x) = \int_{|v|\geq\lambda} e^{-|v|^{m-1}t\Delta}\chi_0(x,v) + \int_0^t \int_{|v|\geq\lambda} e^{-|v|^{m-1}\Delta(t-r)}\partial_v q(r,x,v).$$

- Heat kernel estimates: For $\alpha < 1$,

$$\|u^2\|_{L^1_t H^{2\alpha,1}_x} \lesssim \lambda^{-1-\alpha(m-1)} \|q\|_{\mathscr{M}_{t,x,v}}.$$

- Test case: $m = 1$, $\alpha = 1$, get $u \in L^1_t W^{1,1}_x$.

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- Singular moments

$$\|u^2\|_{L^1_t H^{2\alpha,1}_x} \lesssim \lambda^{-1+\gamma-\alpha(m-1)}\||v|^{-\gamma}q\|_{\mathcal{M}_{t,x,v}}.$$

- For $r = 1$, $\alpha = 1-\varepsilon$: Gives $(u = u^1 + u^2)$

$$u \in (L^1_{t,x}, L^1_t H^{2\alpha,1}_x)_{\theta,\infty} \subseteq L^1_t W^{\frac{2}{m}-\varepsilon,1}_x.$$

Singular moments:

- Recall

$$\partial_t \chi = |v|^{m-1}\Delta_x \chi + \partial_v q \text{ on } (0,T)\times\mathbb{R}^d_x \times \mathbb{R}_v,$$

- Multiplying with $v^{[1-\gamma]}$ and integrating yields

$$\partial_t \underbrace{\int_{v,x} \chi v^{[1-\gamma]}}_{=\frac{1}{2-\gamma}\int_x |u|^{2-\gamma}} = \int_{v,x} v^{[1-\gamma]}|v|^{m-1}\Delta_x \chi + \int_{v,x} v^{[1-\gamma]}\partial_v q$$

$$= -(1-\gamma)\int_{v,x} |v|^{-\gamma} q.$$

Lemma

For $\gamma \in (-\infty, 1)$ we have

$$\|u(t)\|^{2-\gamma}_{L^{2-\gamma}(\mathbb{R}^d_x)} + (2-\gamma)(1-\gamma)\int_0^t \int_{v,x} |v|^{-\gamma} q \le \|u(0)\|^{2-\gamma}_{L^{2-\gamma}(\mathbb{R}^d_x)}.$$

Recall

- So far
$$u \in (L^1_{t,x}, L^1_t H^{2\alpha,1}_x)_{\theta,\infty} \subseteq L^1_t W^{\frac{2}{m}-\varepsilon,1}_x.$$
- Know: For all $r \geq 1$,
$$\|u^1\|_{L^r_{t,x}} = \left\| \int_{|v|\leq\lambda} \chi(u(t,x),v) \right\|_{L^r_{t,x}} \lesssim \lambda$$
$$\|u^2\|_{L^1_t H^{2\alpha,1}_x} \lesssim \lambda^{-1+\gamma-\alpha(m-1)} \| |v|^{-\gamma} q \|_{\mathcal{M}_{t,x,v}}.$$
- Real interpolation: Problem $L^1_t H^{2\alpha,1}_x \hookrightarrow L^r_{t,x}$ only if $r=1$, otherwise $(L^r_{t,x}, L^1_t H^{2\alpha,1}_x)_{\theta,\infty}$ not controlled.
- No optimal integrability.

- Idea: Micro-local decomposition of the Fourier-space depending on the degeneracy in $|v|^{m-1}$.
- Aim: Micro-local decomposition is chosen so that all regularity is on $\tilde{u}^0$, while $\tilde{u}^1$ is only $L^1_{t,x}$:

$$u \in (\underbrace{L^r_t H^{2\alpha,r}_x}_{\ni \tilde{u}^0}, \underbrace{L^1_{t,x}}_{\ni \tilde{u}^1})_{\theta,\infty} \subseteq L^{m-\varepsilon}_t W^{\frac{2}{m}-\varepsilon, m-\varepsilon}_x .$$

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- Recall: Kinetic form for $\chi(u(t,x),v) = 1_{v<u(t,x)} - 1_{v<0}$

$$\partial_t \chi = |v|^{m-1}\Delta\chi + \partial_v q \text{ on } (0,T)\times\mathbb{R}^d_x\times\mathbb{R}_v,$$

for some $q \in \mathscr{M}^+$.
- Fourier transformation in time and space (modulo cut-off in time)

$$\underbrace{i\tau\hat{\chi} - |v|^{m-1}|\xi|^2\hat{\chi}}_{=:\mathscr{L}(i\tau,\xi,v)\hat{\chi}} = \partial_v \hat{q}.$$

- Hence, informally,

$$\hat{\chi} = \frac{1}{i\tau - |v|^{m-1}|\xi|^2}\partial_v\hat{q} = \frac{1}{\mathscr{L}(i\tau,\xi,v)}\partial_v\hat{q}.$$

- Gain regularity, depending on the degeneracy of the operator $\mathscr{L}(i\tau,\xi,v)$.

- Micro-local decomposition:

$$\phi_0(\xi) + \sum_{j\geq 1} \phi_1(2^{-j}\xi) = 1.$$

Decompose χ by

$$\hat{\chi} = \underbrace{\phi_0\left(\frac{\mathscr{L}(i\tau,\xi,v)}{\delta}\right)\hat{\chi}}_{\chi^0} + \sum_{j\geq 1} \underbrace{\phi_1\left(\frac{\mathscr{L}(i\tau,\xi,v)}{2^j\delta}\right)\hat{\chi}}_{\chi^1_j}.$$

- Paley-Littlewood decomposition (in space) to work on fixed blocks of Fourier modes.
- On non-degenerate parts use the equation ($\hat{\chi} = \frac{1}{\mathscr{L}(i\tau,\xi,v)}\partial_v \hat{q}$) and velocity-average.
- Establish multiplier estimates to control regularity of χ^0.

Obstacles:

1. Integrability: Established methods yield good estimates only in an L^2-framework. This prevents from obtaining optimal integrability exponents
   –> Introduce a new notion of isentropic truncation properties for Fourier multipliers.
2. Established methods can only make use of the fact that q has finite mass. This necessarily leads to sub-optimal estimates.
   –> Solution: Use that q allows singular moments $\int |v|^{-1+} dq < \infty$.
3. Bootstrapping: Established methods rely on bootstrapping, i.e. assuming that $u \in W_x^{\alpha,1}$ for some α use that $\chi(u) \in W_{x,v}^{\alpha,1}$. But: This is true for $\alpha \leq 1$ only!

Definition (Isotropic truncation property)

Let $m:\mathbb{R}^d_\xi \times \mathbb{R}_v \to \mathbb{C}$ isotropic in ξ. Then m satisfies the isotropic truncation property if for every bump ϕ_0 supported on a ball in $\mathbb{C}$, every bump ϕ_1 supported in $\{\xi \in \mathbb{C} : 1 \le |\xi| \le 4\}$ and every $1 < p < \infty$

$$M_{\phi_0,J} f(x,v) := \mathscr{F}_x^{-1} \phi_1\left(\frac{|\xi|^2}{J^2}\right) \phi_0\left(\frac{m(\xi,v)}{\delta}\right) \mathscr{F}_x f(x)$$

is an L^p_x-multiplier for all $v \in \mathbb{R}$, $J = 2^j, j \in \mathbb{N}$ and, for all $r \ge 1$,

$$\Big\| \|M_{\phi_0,J}\|_{\mathscr{M}^p} \Big\|_{L^r_v} \lesssim |\Omega_m(J,\delta)|^{\frac{1}{r}},$$

where

$$\Omega_m(J,\delta) := \{v \in \mathbb{R} : |\frac{m(J,v)}{\delta}| \in \mathrm{supp}\phi_0\}.$$

Example: $\mathscr{L}(\xi,v) = -|\xi|^2 b(v)$, for $b:\mathbb{R}\to\mathbb{R}_+\cup\{0\}$ being measurable.
Proof: Hörmander–Mihlin Multiplier Theorem

Assume quantitative non-linearity

$$|\Omega_m(J,\delta)| := |\{v \in \mathbb{R} : |\frac{m(J,v)}{\delta}| \in \mathrm{supp}\phi_1\}| \le (\frac{\delta}{J^\beta})^\alpha.$$

Then, with isotropic truncation property,

$$\begin{aligned}
\|\int \chi_J^0\, dv\|_{L^p_{t,x}} &= \|\int \mathscr{F}^{-1}_{t,x}\phi_1(\frac{\xi}{J})\phi_0\left(\frac{\mathscr{L}(i\tau,i\xi,v)}{\delta}\right)\mathscr{F}_{t,x} f^0\phi\, dv\|_{L^p_{t,x}} \\
&\le \int \|\mathscr{F}^{-1}_{t,x}\phi_1(\frac{\xi}{J})\phi_0\left(\frac{\mathscr{L}(i\tau,i\xi,v)}{\delta}\right)\mathscr{F}_{t,x} f^0\phi\|_{L^p_{t,x}}\, dv \\
&\lesssim \int \|\mathscr{F}^{-1}_{t,x}\phi_1(\frac{\xi}{J})\phi_0\left(\frac{\mathscr{L}(i\tau,i\xi,v)}{\delta}\right)\mathscr{F}_{t,x}\|_{\mathscr{M}^p}\|f^0\phi\|_{L^p_{t,x}}\, dv \\
&\le \left\|\|\phi_1(\frac{\xi}{J})\phi_0\left(\frac{\mathscr{L}(i\tau,i\xi,v)}{\delta}\right)\|_{\mathscr{M}^p}\right\|_{L^r_v}\|f^0\phi\|_{L^{r'}_v L^p_{t,x}} \\
&\lesssim (\frac{\delta}{J^\beta})^{\frac{\alpha}{r}}\|\chi^0\|_{L^{r'}_v L^p_{t,x}}.
\end{aligned}$$

Note: Use $\chi \in L^1_{t,x,v} \cap L^\infty_{t,x,v}$.

$\rightarrow$ Optimal regularity estimate regarding differentiability and integrability.

Optimal regularity for the degenerate parabolic Anderson model

- The degenerate parabolic Anderson model:

$$\partial_t u = \Delta u^{[m]} + u\xi \quad \text{on } (0,T)\times I$$

  with $u_0 \in L^1(I)$, ξ spatial white noise, $I \subseteq \mathbb{R}$ bounded interval and zero Dirichlet boundary conditions.
- Note: $\xi \in C^{-1/2-} = B^{-1/2-}_{\infty,\infty}$.

Corollary

Let $u_0 \in L^{m+1}(I)$. Then there exists a weak solution u satisfying, for all $p \in [1,m)$, $s \in [0, \frac{3}{2}\frac{1}{m})$,

$$u \in L^p([0,T]; W^{s,p}_{loc}(I)),$$

with, for all $T \geq 0$, $\mathcal{O} \subset\subset I$,

$$\|u\|_{L^p([0,T];W^{s,p}(\mathcal{O}))} \lesssim \|u_0\|^{m+1}_{L^{m+1}(I)} + \|S\|^{\tau}_{B^{-\eta}_{\infty,\infty}} + 1,$$

for some $\tau \geq 2$ and $\eta \in (\frac{1}{2}, 1]$ small enough.

Space-time optimal regularity for the porous medium equation

What was left open so far:

- Space-*time* regularity
- Initial data in $L^1(\mathbb{R}^d_x) \rightarrow$ application to the Barenblatt solution
- Higher order integrability & non-homogeneous estimates

Theorem (G., Sauer, Tadmor; 2019)

Let $u_0 \in L^1(\mathbb{R}^d) \cap L^\rho(\mathbb{R}^d)$, $S \in L^1([0,T] \times \mathbb{R}^d) \cap L^\rho([0,T] \times \mathbb{R}^d)$ *for some* $\rho \in [1,\infty)$ *and assume* $m \in (1,\infty)$. *Let* u *be the unique entropy solution. Let* $p \in (\rho, m-1+\rho)$ *and define*

$$\kappa_t := \frac{m-1+\rho-p}{p}\frac{1}{m-1}, \quad \kappa_x := \frac{p-\rho}{p}\frac{2}{m-1}.$$

Then

1. *For all* $\sigma_t \in [0,\kappa_t)$ *and* $\sigma_x \in [0,\kappa_x)$ *we have*

$$u \in W^{\sigma_t,p}(0,T;W^{\sigma_x,p}(\mathbb{R}^d)).$$

2. *Let* $s \in [0,1]$ *and define*

$$p := s(m-1)+1, \quad \kappa_t := \frac{1-s}{s(m-1)+1}, \quad \kappa_x := \frac{2s}{s(m-1)+1}.$$

Then for all $\sigma_t \in [0,\kappa_t)$, $\sigma_x \in [0,\kappa_x)$ *and* $q \in [1,p]$ *we have*

$$u \in W^{\sigma_t,q}(0,T;W^{\sigma_x,q}(\mathscr{O})).$$

Difficulties:

- Identifying the right anisotropic fractional spaces
  - Fourier analytic method works nicely for *homogeneous* Besov spaces only
  - Leads to Schmeisser, Triebel's dominating mixed anisotropic Besov spaces
  - Embedding to non-homogeneous, standard Sobolev spaces delicate
- L^1-data: Singular moments $\int |v|^{-\gamma} q$, $\gamma \in (0,1)$ not finite anymore $\rightarrow$ Respect the different source of difficulty at the degeneracy $|v| = 0$ and the singularity at $|v| = \infty$.

Singular moments for the kinetic measure 2:

- Recall

$$\partial_t \chi = |v|^{m-1}\Delta_x \chi + \partial_v q \text{ on } (0,T)\times\mathbb{R}^d_x\times\mathbb{R}_v,$$

- Previous: Multiplying with $v^{[1-\gamma]}$ and integrating.
- Instead: Multiply by $\mathrm{sgn}_+(v-v_0)$ to get

$$\partial_t \underbrace{\int_{v,x} \chi \mathrm{sgn}_+(v-v_0)}_{=\int_x (u-v_0)_+ \le \int_x |u|} = -\int_x q(t,x,v_0).$$

i.e.

$$\sup_{v_0} \int_{t,x} q(t,x,v_0) \le \int_x |u_0|. \qquad (\star)$$

- At $|v|=0$ use $\int_{|v|<K} |v|^{-\gamma} q$, $\gamma\in(0,1)$ is finite. At $|v|=\infty$ use $(\star)$.

Identifying the right spaces

Definition

Let $\sigma_i \in (-\infty, \infty)$, $i = t, x$,

1. The homogeneous Besov space with dominating mixed derivatives $S^{\overline{\sigma}}_{p,\infty}\dot{B}(\mathbb{R}^{d+1})$ is given by

$$S^{\overline{\sigma}}_{p,\infty}\dot{B} := S^{\overline{\sigma}}_{p,\infty}\dot{B}(\mathbb{R}^{d+1}) := \{f \in \mathscr{Z}' : \|f\|_{S^{\overline{\sigma}}_{p,\infty}\dot{B}} < \infty\},$$

   with the norm

$$\|f\|_{S^{\overline{\sigma}}_{p,\infty}\dot{B}} := \sup_{l,j\in\mathbb{Z}} 2^{\sigma_t l} 2^{\sigma_x j} \|\mathscr{F}^{-1}_{t,x} \eta_l \varphi_j \mathscr{F}_{t,x} f\|_{L^p(\mathbb{R}^{d+1})}.$$

Lemma

Let $\sigma_t, \sigma_x > 0$ and $p \in [1, \infty]$. Then

$$\left(L^p(\mathbb{R}^{d+1}) \cap \tilde{L}^p_x \dot{B}^{\sigma_t}_{p,\infty} \cap \tilde{L}^p_t \dot{B}^{\sigma_x}_{p,\infty} \cap S^{\overline{\sigma}}_{p,\infty}\dot{B}\right) = S^{\overline{\sigma}}_{p,\infty} B \subset W^{\kappa_t,p}(\mathbb{R}; W^{\kappa_x,p}(\mathbb{R}^d)),$$

for $\kappa_t < \sigma_t$, $\kappa_x < \sigma_x$.

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