Regularization of long-time asymptotics by noise.

Benjamin Gess[1]

[1]Institut für Mathematik,
Humboldt-Universität zu Berlin

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Outline

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2. The stochastic case
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   - Degenerate SPDE with additive noise
   - Degenerate SPDE with multiplicative noise
   - Finite speed of propagation for SPME

The deterministic case

The deterministic case

The deterministic case

- Non-degenerate PDE (Chafee-Infante equation)

$$dX_t = \Delta X_t dt + (\lambda X_t - X_t^3)dt, \quad \lambda \geq 0. \tag{RDE}$$

vs. degenerate PDE (porous medium equation, $m > 1$)

$$dX_t = \Delta X_t^m dt + \lambda X_t dt, \quad \lambda \geq 0. \tag{PME}$$

For simplicity write: X^m for $|X|^{m-1}$X.

- Both with zero Dirichlet boundary conditions on bounded domains $\mathcal{O} \subseteq \mathbb{R}^d$.

Reaction diffusion equations

- Linearization of Chafee-Infante:

$$dX_t = \Delta X_t dt + \lambda X_t dt$$

- Expansion of X_t in eigenvectors e_k of $-\Delta$: $X_t = \sum_k X_t^k e_k$ yields

$$dX_t^k = (\lambda - \lambda_k) X_t^k dt$$

- For $\lambda = 0$ the invariant solution $X \equiv 0$ is (exponentially) stable.
- As λ increases 0 bifurcates whenever λ crosses eigenvalues λ_k.

- Consequence: Finite dimensional attractor with explicitly known dimension (for Chafee-Infante).

Porous medium equations

- PME:

$$\begin{aligned} dX_t &= \Delta X_t^m dt + \lambda X_t dt \\ &= {X_t}^{m-1} \Delta X_t dt + {X_t}^{m-2} |\nabla X_t|^2 dt + \lambda X_t dt. \end{aligned} \tag{PME}$$

- Intuitively: Ellipticity constant at the invariant solution $X \equiv 0$ degenerates $\rightarrow$ all modes become unstable.
- Technical obstacle: Cannot proceed by linearization. How to get lower bounds?
- [Efendiev, Zelik; 2008]: Attractor of (PME) has infinite fractal dimension for $\lambda > 0$.

The stochastic case

The stochastic case

Stochastic reaction diffusion equation

Case 1: Additive noise

- Consider
$$dX_t = \Delta X_t dt + \lambda X_t dt + dW_t.$$
- Pathwise interpretation:
$$dX_t(\omega) = \Delta X_t(\omega) dt + \lambda X_t(\omega) dt + dW_t(\omega).$$

Stochastic reaction diffusion equation

- Consider

$$dX_t = \Delta X_t dt + \lambda X_t dt + dW_t.$$

- Solution becomes a random path in $L^2(\mathscr{O})$:

$$X : \mathbb{R}_+ \times \Omega \to L^2(\mathscr{O})$$

$\to$ Random dynamical system $\varphi(t,\omega)x = X_t^x(\omega)$ on $L^2(\mathscr{O})$.

- Consider the laws:

$$t \mapsto \mathscr{L}(X_t) \in \mathscr{P}(L^2(\mathscr{O})).$$

Then $\mathscr{L}(X_t)$ becomes a (deterministic) semigroup on $\mathscr{P}(L^2(\mathscr{O}))$.

Stochastic reaction diffusion equation

- Assume $W_t = \sum_{k=1}^{\infty} \sigma_k e_k \beta_t^k$.
- Expansion in eigenmodes: $X_t = \sum_k X_t^k e_k$ yields

$$dX_t^k = (\lambda - \lambda_k) X_t^k dt + \sigma_k d\beta_t^k.$$

- Ergodicity for Chafee-Infante if $\sigma_k \neq 0$ for all $\lambda > \lambda_k$ $\rightarrow$ regularization by noise on the level of ergodicity

$$\mathscr{L}(X_t) \rightarrow_{TV} \mu.$$

Stochastic reaction diffusion equation

Stabilization on the level of the stochastic flow

- Reminder of [Caraballo, Crauel, Langa, Robinson; 2007], related [Chueshov, Scheutzow; 2004].
- RDS given by $\varphi(t,\omega)x = X_t^x(\omega)$.
- Known: X ergodic, φ has a random attractor $\mathscr{A}$.
- $\mathscr{A}$ is bounded in L^∞
  $\rightarrow$ can choose $\underline{a}(\omega) = \inf \mathscr{A}(\omega)$, $\bar{a}(\omega) = \sup \mathscr{A}(\omega)$:

$$\underline{a}(\omega) \le \mathscr{A}(\omega) \le \bar{a}(\omega).$$

- φ is order preserving, i.e. $x \le y$ then $\varphi(t,\omega)x \le \varphi(t,\omega)x$
- Then $\underline{a}, \bar{a}$ are invariant:

$$\varphi(t,\omega)\bar{a}(\omega) = \bar{a}(\theta_t\omega)$$
$$\varphi(t,\omega)\underline{a}(\omega) = \underline{a}(\theta_t\omega)$$

- Hence, $\mathscr{L}(\underline{a}), \mathscr{L}(\bar{a})$ are invariant measures for X_t. By ergodicity $\mathscr{L}(\underline{a}) = \mathscr{L}(\bar{a})$, thus $\underline{a} = \bar{a}$ and

$$\mathscr{A}(\omega) = \{\bar{a}(\omega)\}.$$

Stochastic reaction diffusion equation

- Three main ingredients:
  - Ergodicity of Markovian dynamics.
  - Existence of a random attractor $\mathscr{A}$ bounded in L^∞.
  - Order-preserving property of φ.

Stochastic reaction diffusion equation

Multiplicative Stratonovich noise

- Spatially homogeneous noise:

$$dX_t = \Delta X_t dt + \lambda X_t dt + \sigma X_t \circ d\beta_t$$

Transformation: $Y_t = e^{-\sigma\beta_t} X_t$ solves

$$dY_t = \Delta Y_t dt + \lambda Y_t dt$$

$\rightarrow$ *No* regularization.

- Spatially inhomogeneous noise:

$$dX_t = \Delta X_t dt + \lambda X_t dt + \sum_{k=1}^{\infty} B_k X_t \circ d\beta_t^k,$$

B_k suitable linear operators. It is enough to have B_k acting on the finite-dimensional unstable part. Under proper assumptions on B_k system becomes ergodic.

Aims/Questions

- Analogous questions in the degenerate case:
  - Regularization by additive noise
  - Regularization (or not) by Stratonovich space-time noise

Degenerate SPDE with additive noise

Degenerate SPDE with additive noise
based on [G., JDDE, 2013]

Degenerate SPDE with additive noise

- Consider

$$dX_t = \Delta X_t^m dt + \lambda X_t dt + dW_t, \quad m > 1$$

  with zero Dirichlet boundary conditions on a bounded domain.
- Aim: Capture regularization by noise on the level of the stochastic flow.
- Tasks
  - Construct stochastic flow
  - Prove existence of a random attractor $\mathscr{A}$
  - Show $\mathscr{A}$ is 0-dimensional (Note: Stabilization of infinitely many unstable modes).
- Obstacles:
  - Construction of stochastic flow requires regular noise W_t [Beyn, G., Lescot, Röckner; CPDE, 2011]
  - Ergodicity requires non-degenerate noise [Liu; JEE, 2009]

Degenerate SPDE with additive noise

- RDS to

$$dX_t = \Delta X_t^m dt + \lambda X_t dt + dW_t \tag{SPME}$$

usually is constructed via transformation $Y_t = X_t - W_t$ yields

$$dY_t = \Delta (Y_t + W_t)^m dt + \lambda (Y_t + W_t) dt.$$

Solve pathwise $\rightarrow$ RDS.

- Problem: Only makes sense if W_t takes values in $\mathscr{D}(\Delta(\cdot)^m)$.
- Solution: Construct strictly stationary solution Z_t ($\sim$ stationary, nonlinear Ornstein-Uhlenbeck process) to strongly monotone part of (SPME), i.e. to

$$dX_t = \Delta X_t^m dt + dW_t.$$

Exists by dissipativity method.

- Transform (SPME) by $Y_t = X_t - Z_t$ into a random PDE. Advantage: Z_t takes values in $\mathscr{D}(\Delta(\cdot)^m)$.
- Result: Existence of RDS for trace-class noise W_t.

Degenerate SPDE with additive noise

Assume that for each compact set $K \subseteq V$ there is an interval $[x,y] \subseteq H$ such that $K \subseteq [x,y]$.

Theorem

Let φ be an order preserving, strongly mixing RDS with ergodic measure μ concentrated on V. Then there exists a unique, minimal weak point random attractor $\mathscr{A}(\omega) = \{v(\omega)\}$ given by a single random point $v : \Omega \to H$ with $v \in V$, $\mathbb{P}$-a.s..

Degenerate SPDE with multiplicative noise

Degenerate SPDE with multiplicative noise
based on [G., SIMA (to appear), 2013]

Degenerate SPDE with multiplicative noise

- Recall: Reaction-Diffusion equations may be stabilized by space-time Stratonovich noise.
- Consider:

$$dX_t = \Delta X_t^m dt + \lambda X_t dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^k. \qquad \text{(SPME)}$$

- Recall that in the deterministic case ($f_k \equiv 0$, $\lambda > 0$) the attractor has infinite fractal dimension.
- Question: May we still stabilize (SPME) by multiplicative space-time Stratonovich noise?
- Aim: We show that the random attractor for (SPME) has infinite fractal dimension, i.e. no stabilization occurs.

Construction of an RDS

- Recall:

$$dX_t = \Delta X_t^m dt + \lambda X_t dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^{(k)}.$$

- Set $Y_t := e^{\mu_t} X_t$, where $\mu_t = -\lambda t - \sum_{k=1}^{N} f_k \beta_t^{(k)}$. Then

$$\partial_t Y(t,x) = e^{\mu(t,x)} \Delta \left(e^{-\mu(t,x)} Y(t,x) \right)^m.$$

- Existence and uniqueness in [Barbu, Röckner, JDE, 2011], [G., AoP (to appear), 2013].

Degenerate SPDE with multiplicative noise

Idea of the proof:

- The invariant solution 0 is unstable, has ∞-dimensional unstable manifold.

$$\mathscr{M}^+(0,\omega) := \{u_0 \in X \mid \exists u : (-\infty,0] \to X, \text{ such that } \varphi(t;\theta_{-t}\omega)u(-t) = u_0 \\ \text{for all } t \geq 0 \text{ and } \|u(t)\|_{L^\infty(\mathscr{O})} \to 0 \text{ for } t \to -\infty\}.$$

  (Recall: Cannot proceed via linearization)

- In terms of the transformed equation: Find $v \in L^\infty((-\infty,0] \times \mathscr{O})$ such that

$$u_0 = Y(0,-t;\omega)v(-t), \quad \forall t \geq 0. \qquad (*)$$

Degenerate SPDE with multiplicative noise

- Use time transformation to reduce to finite time interval: $G(t) = \log(t) : (0,1] \mapsto (-\infty, 0]$. Get

$$\partial_t U = e^{\mu_{G(t)} + \eta G(t)} \Delta (e^{-\mu_{G(t)} + \eta G(t)} U_t)^m, \text{ on } [0,1] \times \mathscr{O}. \qquad (**)$$

  $\rightarrow$ L^∞ Solutions to (**) give solutions to (*), i.e. elements $u_0 = v(0)$ of $\mathscr{M}^+(0,\omega)$.

- Need to know that solutions obtained this way have sufficiently different end-value $v(0)$.
  $\rightarrow$ prove finite speed of propagation for (**).

Finite speed of propagation for SPME

Finite speed of propagation for SPME

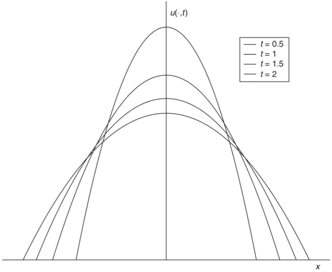
Finite speed of propagation for SPME

- Recall:

$$dX_t = X_t^{m-1}\Delta X_t + {X_t}^{m-2}|\nabla X_t|^2. \qquad \text{(Det. PME)}$$

  The diffusivity coefficient vanishes for $X_t \to 0$.

- Limited regularity: ∇X_t discontinuous. E.g. Barenblatt solutions:



- Regularity is limited precisely at the free boundary.

Known results

- *Finite speed of hole-filling* [Barbu, Röckner, EJP 2012]:
  Let X_0 vanish in $B_R(x_0)$. Then X vanishes in $B_{R(t,\omega)}(x_0)$ for some function $R(\cdot,\omega):[0,T]\to(0,R)$.
- No uniform control on $R(t,\omega)$ in x_0 $\to$ cannot deduce finite speed of propagation
- No information about optimality of the bounds

Basic idea of proof

- Recall:

$$\partial_t Y(t,x) = e^{\mu(t,x)} \Delta \left(e^{-\mu(t,x)} Y(t,x) \right)^m.$$

- freeze coefficients in space:

$$\partial_t Y(t,x) \approx e^{\mu(t,x_0)} \Delta \left(e^{-\mu(t,x_0)} Y(t,x) \right)^m,$$

on small balls $B_r(x_0)$.

Theorem

Let X be a bounded, non-negative solution to (SPME). Then,

$$supp(X_t) \subseteq B_{\sqrt{t}\left(\frac{H^{m-1}}{C_{det}}\right)^{\frac{1}{2}} \sqrt{C_t(\omega)}}(supp(X_0)), \quad \forall t \in [0,T],$$

with $C_t \to 1$ for $t \to 0$, $H = \|X\|_\infty$.

Degenerate SPDE with multiplicative noise

- Back to dimension of the random attractor for:

$$dX_t = \Delta X_t^m dt + \lambda X_t dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^{(k)}.$$

- By finite speed of propagation we get:

Theorem

The Kolmogorov ε-entropy of $\mathscr{A}$ is bounded below by

$$\mathbb{H}_\delta(\mathscr{A}(\omega)) \geq C(\omega)\delta^{\frac{-d(m-1)}{2+d(m-1)}}, \quad \forall \omega \in \Omega,$$

In particular, the fractal dimension $d_f(\mathscr{A}(\omega))$ is infinite for all $\omega \in \Omega$.