REGULARIZATION AND WELL-POSEDNESS BY NOISE FOR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

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Dedicated to Michael Röckner in honor of his 60th birthday.

Abstract. We give a brief introduction and overview of the topic of regularization and well-posedness by noise for ordinary and partial differential equations. The article is an attempt to outline in a concise fashion different directions of research in this field that have attracted attention in recent years. We close the article with a look on more recent developments in the field of nonlinear SPDE, focusing on stochastic scalar conservation laws and porous media equations. The article is tailored at master/PhD level, trying to allow a smooth introduction to the subject and pointing at a large list of references to allow further in-depth study.

1. Introduction

1.1. **Finite dimensional case.** In this section we will briefly recall some well-posedness by noise results for the case of ordinary differential equations. Since the available literature is vast and since we are mostly interested in the case of (S)PDE, we will restrict to pointing out some basic examples and a few selected results.

The classical Cauchy-Lipschitz Theorem yields that ordinary differential equations

$$dX_t^x = b(t, X_t^x)dt, \quad X_0^x = x \in \mathbb{R}^d. \tag{1.1}$$

with $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, have a unique global solution, provided that b is regular enough. For example, in a standard first course in analysis the well-posedness of solutions to (1.1) is usually proven assuming that b is locally Lipschitz continuous and of sublinear growth, that is, for some constant $C \geq 0$,

$$|b(t,x)| \leq C(1+|x|) \quad \text{for all } t \geq 0, x \in \mathbb{R}^d. \tag{1.2}$$

There are several possibilities to relax these assumptions. For example, the growth condition (1.2) may be replaced by a one-sided growth condition

$$(b(t,x), x) \leq C(1+|x|) \quad \text{for all } t \geq 0, x \in \mathbb{R}^d \tag{1.3}$$

and (local) Lipschitz continuity can be replaced, to some extent, by (local) one-sided Lipschitz continuity, that is, for some constant $C \geq 0$ (continuous function $C : \mathbb{R}^d \to \mathbb{R}_+$ resp.),

$$(b(t,x) - b(t,y), x-y) \leq C(y)|x-y|^2 \quad \text{for all } x, y \in \mathbb{R}^d. \tag{1.4}$$

Such type of extensions are of particular importance in the case of (S)PDE, leading to the notion of (locally) monotone operators (cf. e.g. [10, 83] and the references therein). In fact, the Cauchy-Lipschitz Theorem provides more information, namely

Date: April 13, 2017.
2000 *Mathematics Subject Classification.* 60H15, 35R60, 35L65.
Key words and phrases. regularization and well-posedness by noise, stochastic scalar conservation laws, stochastic porous medium equation, stochastic Hamilton-Jacobi equation.

the existence and uniqueness of a continuous flow of solutions $\phi : [0,T] \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\phi_{s,t} = \phi_{r,t} \circ \phi_{s,r}, \quad \phi_{s,s} = Id \quad \forall s \le r \le t,\ s,r,t \in [0,T] \tag{1.5}$$

and $\phi_{s,t}(x) = X_t^x$ is the unique solution to (1.1) started in x at time $s \in [0,T]$. These classical results were extended in the seminal work [43] under less restrictive assumptions on the drift b, more precisely, assuming

$$\frac{|b|}{1+|x|} \in L^1([0,T]; L^\infty(\mathbb{R}^d)) + L^1([0,T]; L^1(\mathbb{R}^d)), \tag{1.6}$$

and

$$b \in L^1_{loc}([0,T]; W^{1,1}_{loc}(\mathbb{R}^d)) \text{ and } \operatorname{div} b \in L^1([0,T]; L^\infty(\mathbb{R}^d)). \tag{1.7}$$

Under these assumptions it was shown in [43] that there is a unique map $\phi : [0,T] \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying (1.5) a.e. on $\mathbb{R}^d$ and for all $s \in [0,T]$,

$$\frac{d}{dt}\eta(\phi_{s,t}(x)) = D\eta(\phi_{s,t}(x)) \cdot b(t, \phi_{s,t}(x)), \quad \eta(\phi_{s,s}(x)) = \eta(x)$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$, η sufficiently smooth and integrable and

$$e^{-|A(t)-A(s)|}\lambda \le (\phi_{s,t})_*\lambda \le e^{-|A(t)-A(s)|}\lambda \tag{1.8}$$

for some constant $C_0 \ge 0$, where A is absolutely continuous, $A(0) = 0$ and nondecreasing, λ denotes the Lebesgue measure on $\mathbb{R}^d$ and $(\phi_{s,t})_*\lambda$ the push-forward of λ under $x \mapsto \phi_{s,t}(x)$. In fact, in the construction of solutions the function A is given by $A(t) := \int_0^t \|\operatorname{div} b(r,\cdot)\|_{L^\infty(\mathbb{R}^d)} dr$. The proof of [43] relies on the interpretation of (1.1) as the system of characteristics associated to the linear transport equation

$$du + b(t,x) \cdot \nabla u dt = 0, \quad \text{on } [0,T] \times \mathbb{R}^d, \tag{1.9}$$

that is, on the relation,

$$u(t,x) = u_0(\phi_{0,t}^{-1}(x)),$$

In the sequel of this article this relation between finite dimensional ODE and linear transport equations will be exploited several times. The results of [43] have been generalized in [1] where (1.7) were relaxed to

$$b \in L^1_{loc}([0,T]; BV_{loc}(\mathbb{R}^d)) \text{ and } \operatorname{div} b \in L^1([0,T]; L^\infty(\mathbb{R}^d)), \tag{1.10}$$

with several subsequent generalizations, cf. e.g. [26] and [2] for a nice survey.

On the other hand, relaxing the bounded divergence condition on b can lead to simple counter-examples, a standard one being $b(x) = \operatorname{sgn}(x)\sqrt{|x|}$. Clearly, Lipschitz continuity of b fails at zero, causing non-uniqueness of solutions. Indeed, for each given $T \ge 0$,

$$X_t^0 := \begin{cases} 0 & \text{for } t \in [0,T] \\ \pm\frac{1}{2}(t-T)^2 & \text{for } t \ge T \end{cases} \tag{1.11}$$

defines a solution with initial condition $X_0 = 0$. Hence, (forward) uniqueness of solutions fails (branching), but backward uniqueness is satisfied. Another example is given by $b(x) = \sqrt{|x|}$. In this case, solutions are not unique for negative initial conditions x, indeed, for each $T > 0$, setting $c = \sqrt{-x}$,

$$X_t^x := \begin{cases} -\frac{1}{4}(t-2c)^2 & \text{for } t \in [0,2c] \\ 0 & \text{for } t \in [2c, 2c+T] \\ \frac{1}{4}(t-T-2c)^2 & \text{for } t \ge 2c+T \end{cases} \tag{1.12}$$

defines a solution to (1.1) with $b(x) = \sqrt{|x|}$. In this example, we observe both non-uniqueness of solutions (branching) as well as collisions of solutions (coalescence). In this sense both forward and backward uniqueness of solutions are violated for (1.12). As a last example, consider $b(x) = -\mathrm{sgn}(x)\sqrt{|x|}$. In this case, with $c = \sqrt{|x|}$, solutions are given by

$$X_t^x := \mathrm{sgn}(x) \begin{cases} \frac{1}{4}(t-2c)^2 & \text{for } t \in [0,2c] \\ 0 & \text{for } t \geq 2c. \end{cases} \tag{1.13}$$

Hence, forward uniqueness is satisfied (no branching), while backward uniqueness is not. Several more counter-examples may be found in [43, 50]. In particular, in [43] it has been shown that the conditions (1.6), (1.10) are essentially sharp and uniqueness of solutions can fail if one of the conditions is "significantly" weakened.

This dramatically changes if the system (1.1) is perturbed by non-degenerate noise. The simplest case of well-posedness by noise can be observed in the case of finite dimensional stochastic differential equations perturbed by additive Brownian noise, that is, for $\sigma > 0$,

$$dX_t^x = b(t, X_t^x)dt + \sigma d\beta_t, \quad X_0^x = x \in \mathbb{R}^d, \tag{1.14}$$

where β denotes a standard Brownian motion in $\mathbb{R}^d$. Classical methods relying on the Girsanov transformation (cf. e.g. [69, 98]) can be used to prove the existence and weak uniqueness for weak solutions to (1.14) if b is only measurable and bounded. In [107] this was significantly strengthened by proving pathwise uniqueness for (1.14) and thus, via the Yamada-Watanabe theorem the existence of strong solutions assuming $b \in L^\infty([0,T] \times \mathbb{R}^d)$. A direct approach to the construction of strong solutions under the same assumption on b has been given in [87, 88], based on Malliavin calculus. A further extension was obtained in [73] where pathwise uniqueness was shown assuming that b satisfies the Krylov-Röckner condition, also sometimes called the strong Ladyzhenskaya-Prodi-Serrin (LPS) condition, that is, $b \in L^q([0,T]; L^p(\mathbb{R}^d))$ with $\frac{d}{p} + \frac{2}{q} < 1$, $p, q \geq 2$. In these works, however, no continuous dependence of the solutions X_t^x on the initial datum x was shown. This was resolved more recently in [45], again under the assumption that b satisfies the Krylov-Röckner condition. Under the same condition, in [9] the Malliavin differentiability of the solution X_t^x was shown. The natural follow-up question if the equivalence to the Lebesgue measure (1.8) is satisfied under these assumptions in some sense is positively answered in [84]. In addition, the important contribution [52] provides the existence of an associated stochastic flow ϕ and estimates on the derivative of the flow $D_x\phi$ for drifts $b \in L^\infty([0,T]; C_b^\alpha(\mathbb{R}^d))$, $\alpha \in (0,1)$, with significant impact on the well-posedness of stochastic transport equations as we shall see below. In [89] these results have been partially extended by proving Sobolev differentiability of the stochastic flow ϕ in the sense that $\phi \in L^2(\Omega; W^{1,p}(\mathbb{R}^d; w))$, where $W^{1,p}(\mathbb{R}^d; w)$ are weighted Sobolev spaces with weight w satisfying an integrability condition, assuming only $b \in L^\infty([0,T] \times \mathbb{R}^d)$.

One approach to prove weak uniqueness of solutions to (1.14) is based on the analysis of the associated Fokker-Planck-Kolmogorov equation, satisfied by the evolution of the law, for which the regularizing effect of the noise becomes apparent. More precisely, at least informally, the law $u(t,x) := \mathrm{Law}(X_t^x)$ satisfies the Fokker-Planck-Kolmogorov equation

$$\partial_t u = \frac{\sigma^2}{2}\Delta u + \mathrm{div}(bu), \quad \text{on } [0,T] \times \mathbb{R}^d, \tag{1.15}$$

where the strongly elliptic term $\frac{\sigma^2}{2}\Delta$ is due to the random perturbation in (1.14). The analysis of Fokker-Planck-Kolmogorov equations and their implication on the well-posedness of SDE has attracted much interest in recent years and many results can be found in the recent monograph [15] and the references therein. An extension of the DiPerna-Lions-Ambrosio theory for conservation equations to Fokker-Planck-Kolmogorov equations (1.15) and its consequences for weak/martingale solutions for SDE with multiplicative noise

$$dX_t^x = b(t, X_t^x)dt + \sigma(t, X_t^x)d\beta_t, \quad X_0^x = x \in \mathbb{R}^d. \tag{1.16}$$

has been put forward in [49, 76, 104]. A uniqueness result for (1.16) relying on a form of local one-sided Lipschitz condition, related to (1.4) can be found in [99]. The attempt of proving (weak) uniqueness of solutions for SPDE via the associated Fokker-Planck-Kolmogorov equations also seems to be a very promising approach in the theory of SPDE. For some references see Section 2 below.

Further extensions of the results recalled above, concerning the question of SDE driven by jump noise have been considered in [65, 96, 97]. Extensions to fractional Brownian motion can be found in [9, 20].

We have already encountered different notions of uniqueness of solutions to (1.14). Namely, weak and pathwise uniqueness. Roughly speaking, pathwise uniqueness means that any two strong solutions X_t^x, Y_t^x to (1.14) coincide $\mathbb{P}$-almost surely. Here, the corresponding $\mathbb{P}$-zero set on which these solutions possibly do not coincide is allowed to depend on the initial condition x. This leads to the notion of path-by-path uniqueness. Roughly speaking, we say that (1.14) satisfies path-by-path uniqueness if there is a set of full measure $\Omega_0 \subseteq C([0,T];\mathbb{R}^d)$ with respect to the Wiener measure such that for each $\omega \in \Omega_0$ solutions to

$$X_t^x = X_0^x + \int_0^t b(r, X_r^x)dr + \omega_t, \quad X_0^x = x \in \mathbb{R}^d \tag{1.17}$$

are unique. Thus, in contrast to pathwise uniqueness, the exceptional zero set is not allowed to depend on the initial datum x in this case. Path-by-path uniqueness for (1.14) for drifts b being measurable and bounded was shown in [36] (cf. [37] for an extension to multiplicative noise). More recently, this was extended in [9] for drifts b satisfying the LPS condition, that is, $b \in L^q([0,T];L^p(\mathbb{R}^d))$ with $\frac{d}{p} + \frac{2}{q} \le 1$, $p, q \ge 2$. See Section 2 for a recent extension of the result of [36] to the setting of SPDE.

In view of this result, a natural question to ask is which properties of a path $\omega \in C([0,T];\mathbb{R}^d)$ lead to well-posedness of (1.17). In other words, to give a characterization of such paths in Ω_0. In this regard, the notion of an irregular path has been introduced in [20], where it is shown that assuming ω to be irregular in their sense implies the uniqueness of solutions to (1.17) under appropriate assumptions on b.

In conclusion, many details of the effect of well-posedness by noise for finite dimensional differential equations are quite well-understood by now. As we will see in the following, this changes drastically when we pass to the infinite dimensional case, that is, when considering the effect of well-posedness by noise for partial differential equations. At the same time, the uniqueness of solutions to nonlinear PDE arising in fluid dynamics is one of the key open questions in PDE theory. The hope to obtain uniqueness at least in the stochastically perturbed situation can be seen as one of the driving forces in the development of the field of SPDE.

1.2. **Linear SPDE.** In view of the success of perturbing ODE by additive noise, recalled in the last section, it is a natural idea to try to obtain analogous effects in the case of PDE. Indeed, in several instances, non-degenerate additive noise has been proven to produce regularizing or even well-posedness effects. Typically, these equations are of semilinear type with a leading non-degenerate, linear operator (cf. (2.1), (2.2) below). On the other hand, the phenomenon of turbulence in fluid dynamics is strongly related to vanishing viscosity, i.e. to hyperbolic systems such as the incompressible Euler equations

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u &= 0 \\ \operatorname{div} u &= 0. \end{aligned}$$

In a more recent line of developments it has been uncovered that in the case of hyperbolic (linear) PDE, linear transport noise, rather than additive noise, induces both regularizing and well-posedness effects. In view of the (informal) relation to non-uniqueness and turbulence in fluid dynamics, these results have attracted a lot of interest.

More precisely, in a series of works Flandoli, Gubinelli, Priola and coworkers [7, 46, 50–52, 54] have considered stochastic transport equations of the type, for $\sigma > 0$,

$$du + b(t,x) \cdot \nabla u dt + \sigma \nabla u \circ d\beta_t = 0, \quad \text{on } [0,T] \times \mathbb{R}^d, \tag{1.18}$$

where β is an $\mathbb{R}^d$-valued standard Brownian motion. The inclusion of noise of the above type corresponds to a random perturbation of the transport term b at the level of the characteristics, that is, the system of characteristics informally corresponding to (1.18) is given by

$$dX_t = b(t, X_t)dt + \sigma d\beta_t, \tag{1.19}$$

thus underlining the close relation to the results recalled in the last section. This viewpoint was adopted in [52] in order to prove that (essentially) bounded, weak solutions to (1.18) are unique as long as $b \in L^\infty([0,T]; C_b^\alpha(\mathbb{R}^d))$ and $\operatorname{div} b \in L^p([0,T] \times \mathbb{R}^d)$ for some $p > 2$. In fact, a good part of the proof in [52] consists in showing the existence of a $C^{1+\alpha'}$-flow of solutions $\phi_{0,t}(x,\omega)$ to (1.19), for $0 < \alpha' < \alpha$, and a bound on the corresponding Jacobian $D\phi_{0,t}(\omega, x)$ in $L^2([0,T]; W^{1,2}_{loc}(\mathbb{R}^d))$ a.s.. Via the representation formula

$$u(t,x) = u_0(\phi_{0,t}^{-1}(x)) \tag{1.20}$$

this immediately implies the existence of a bounded, weak solution to (1.18) for each $u_0 \in L^\infty(\mathbb{R}^d)$. Based on the estimate on the Jacobian also a commutator estimate in the spirit of [43] is shown leading to the uniqueness of weak solutions to (1.18). Note that the conditions imposed on b in [52] improve the ones from the deterministic case (cf. (1.7)). In particular, the model example (1.11) provides an example for which the deterministic transport equation (1.9) does not satisfy uniqueness of weak solutions, whereas the stochastically perturbed version (1.18) does.

Also the results obtained in [46] have a nice reinterpretation in terms of (1.18): Based on the representation formula (1.20) one can see that noise prevents the occurrence of shocks, that is, if $b \in L^q([0,T]; L^p(\mathbb{R}^d, \mathbb{R}^d))$, $\frac{d}{p} + \frac{2}{q} < 1$, $p, q \geq 2$ and $u_0 \in \bigcap_{p \geq 1} W^{1,p}(\mathbb{R}^d)$ then

$$\mathbb{P}\left(u(t,\cdot) \in \bigcap_{p\geq 1} W^{1,p}_{loc}(\mathbb{R}^d) \right) = 1 \quad \text{for all } t \geq 0.$$

As we have recalled above, the arguments introduced in [52] heavily rely on the representation of solutions via stochastic characteristics (1.20). This simple representation is lost for nonlinear PDE. In [7], an alternative argument was proposed under different assumptions on b: Let u be a bounded, weak solution to (1.18). Informally, using Itô's formula yields

$$d\eta(u) + b \cdot \nabla \eta(u) dt = -\nabla \eta(u) \circ d\beta_t, \tag{1.21}$$

for all $\eta \in C^1(\mathbb{R})$. We say that a weak solution to (1.18) is renormalized, if $\eta(u)$ satisfies (1.21) for all $\eta \in C^1(\mathbb{R})$. Hence, under appropriate assumptions on b we expect that weak solutions to (1.18) are renormalized. Indeed, this is justified in [7] under the (classical) assumption $b \in L^1([0,T]; BV_{loc}(\mathbb{R}^d))$. Rewriting the Stratonovich integral in the Itô sense and taking expectation then gives

$$d\mathbb{E}\eta(u) + b \cdot \nabla \mathbb{E}\eta(u)dt = \Delta \mathbb{E}\eta(u)dt.$$

Due to the non-degenerate, parabolic structure of this PDE this implies the uniqueness of bounded, weak solutions to (1.18), assuming only $b \in L^2([0,T]; L^\infty(\mathbb{R}^d)) \cap L^1([0,T]; BV_{loc}(\mathbb{R}^d))$ with $\int_0^T \int_{\mathbb{R}^d} \frac{|\mathrm{div} b|}{(1+|x|)^{N_0}} dxdt < \infty$ for some $N_0 \in \mathbb{N}$.

An extension of the above mentioned results to drifts b being only bounded and measurable has been given in [89]. The results on well-posedness of weak solutions by noise have been furthermore complemented in [46,54,89] by proving better regularity properties (even obtaining *classical* solutions in [54]) for solutions to the stochastically perturbed PDE than in the deterministic case, under various assumptions on b. In [9,51,86,91] similar results have been established for the continuity equation that is (1.18) in divergence form (in the sense that noise prevents the concentration of mass).

The proof of well-posedness by noise for (1.18) opens the way to study selection principles based on vanishing noise. That is, one aims to let $\sigma \downarrow 0$ in (1.18) and to prove that the corresponding sequence of solutions u^σ to (1.18) converges to a limit u being a solution to the deterministic problem. In the case of the linear transport equation (1.18) such a selection principle has been studied in [6]. Concerning vanishing noise selection principles for SDE we refer to [16,41,95,103].

Based on these result one may hope for a similar effect of well-posedness by noise for PDE appearing in fluid dynamics. However, as pointed out in [50] the underlying reasons for non-uniqueness are quite different. In fact, the following negative results for the non-viscous Burgers equation may be found in [50]: Consider

$$du + \partial_x u^2 + \partial_x u \circ d\beta_t = \quad \text{on } [0,T] \times \mathbb{R}. \tag{1.22}$$

Then, setting $v(t,x) := u(t, x+\beta_t)$ we see that, informally, v is a solution to the deterministic Burgers equation

$$dv + \partial_x v^2 = 0 \quad \text{on } [0,T] \times \mathbb{R}. \tag{1.23}$$

In particular, shocks and non-uniqueness of weak solutions still appear in (1.22). Hence, no well-posedness by noise, nor regularization by noise seems to be present in this case. It will be one purpose of the following sections to analyze the validity of this conclusion and to propose different forms of noise that do lead to improvements by noise in the case of nonlinear PDE.

2. Nonlinear SPDE

As compared to the case of linear (stochastic) partial differential equations, much less is known concerning regularizing effects of noise and well-posedness by noise for nonlinear PDE. Historically, the first results in this direction were obtained in the case of viscous PDE perturbed by additive noise. For example, in [64] reaction diffusion equations perturbed by space time white noise of the type

$$du = \Delta u \, dt + f(u) \, dt + dW_t \quad \text{on } [0,1] \tag{2.1}$$

with Neumann boundary conditions were considered and well-posedness was shown assuming only that f is measurable and satisfies a growth condition. Recently, this result was partially sharpened in [18] to path-by-path uniqueness for (2.1) posed on $\mathbb{R}$ and for f being measurable and bounded, thus extending Davie's result [36] from SDE to SPDE. A further extension of the results obtained in [64] was given in [92] where Malliavin differentiability of the solution to (2.1) was shown under the assumption of f being bounded and measurable.

As mentioned above, one of the key aims in understanding well-posedness by noise effects for nonlinear SPDE is the hope to be able to generate uniqueness of solutions in the case of nonlinear PDE arising in fluid dynamics; the most prominent example being the incompressible 3d Navier-Stokes equations. Despite its relevance and considerable effort, only partial, but highly interesting, results could be obtained in this regard. For example, in [55, 56] the 3d Navier-Stokes equations perturbed by sufficiently non-degenerate additive noise were considered

$$\begin{aligned} du + (u\cdot\nabla)u\,dt + \nabla p\,dt &= \Delta u\,dt + dW_t \\ \operatorname{div} u &= 0. \end{aligned} \tag{2.2}$$

The classical regularity result by Cafarelli-Kohn-Nirenberg [19] states, roughly speaking, that the set of singular points of solutions to the 3d Navier-Stokes equations is a Lebesgue zero set. In contrast, for the stochastically perturbed case (2.2) it was shown in [55] that for each fixed time $t \geq 0$, the set of singular points of the solution $u(t)$ to (2.2) is empty $\mathbb{P}$-a.s.. One should note, however, that the $\mathbb{P}$-zero set in this statement is allowed to depend on the time $t \geq 0$.

In [42] it was shown that noise can prevent the collapse of Vlasov-Poisson point charges. For the deterministic Vlasov-Poisson equation

$$\partial_t f + v\partial_x f + E(t,x)\partial_v f = 0$$

with

$$\rho(t,x) = \int_{\mathbb{R}} f(t,x,v)dv, \quad E(t,x) = \int_{\mathbb{R}} F(x-y)\rho(t,y)dy,$$

where $F \in C(\mathbb{R}\setminus\{0\})$, it is known that measure-valued solutions can develop singularities (cf. [85]). In contrast, in [42] it was shown that in the stochastically perturbed case

$$df + v\partial_x f dt + \Big(E(t,x) + \varepsilon\sum_{k=1}^{\infty}\sigma_k(x)\circ d\beta_t^k\Big)\partial_v f dt = 0,$$

where $\varepsilon > 0$, β^k are independent Brownian motions and σ_k satisfy a non-degeneracy condition, no singularities appear. One should also note the related work [53] proving the prevention of coalescence by noise in point vortex dynamics informally corresponding to stochastic 2D Euler equations. In addition, regularizing effects of noise for stochastic kinetic equations have been found in [47].

As we have recalled in the previous section, one way to prove the well-posedness by noise for SDE relies on considering the associated Fokker-Planck-Kolmogorov equation (1.15). In the case of (S)PDE the associated Fokker-Planck-Kolmogorov equation becomes a PDE on a space of functions with infinitely many independent variables. A considerable amount of work has been done in this direction in recent years. For reference let us refer to the books [21, 27, 34] and the recent works [28–33]. For example, in the recent work [33], using the theory of quasi-regular Dirichlet forms and maximal regularity results for Kolmogorov equations in infinite dimensions, the existence and uniqueness of strong solutions to SPDE

$$dX_t = AX_t dt - \nabla V(X_t)dt + B(X_t)dt + dW_t$$

on Hilbert spaces H was shown, where A is a self-adjoint, negative operator with trace class inverse, $V : H \to \bar{\mathbb{R}}$ is convex, proper, lower-semicontinuous satisfying appropriate bounds on its first and second derivative, $B : H \to H$ is measurable and bounded and W is a cylindrical Wiener process on H.

Several further examples of regularization and well-posedness by noise in nonlinear situations can be found in [50].

2.1. **Scalar conservation laws.** In the following sections we will investigate regularizing and well-posedness effects of noise in the case of (stochastic) scalar conservation laws (SCL). The current section offers some introductory material and a short overview of some available results for deterministic and stochastic conservation laws. Since a full account of the theory is well beyond the scope of this introduction, we will restrict to pointing out some results that seem most relevant for the following sections.

Scalar conservation laws

$$\partial_t u + \mathrm{div} F(t,x,u) = 0, \quad \text{on } [0,T]\times\mathbb{R}^d \tag{2.3}$$

model the convective transport of a scalar quantity u induced by a possibly nonlinear flux $F:[0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^d$. In the following we will consider either the Cauchy problem or periodic data for (2.3) unless specified otherwise. Typical examples are given by the inviscid Burgers equation and the Buckley-Leverett equation. It is well-known that the hyperbolic structure of these equations may cause the existence of singularities, such as shocks and mass concentration. On the other hand, distributional solutions to (2.3) can be shown to be non-unique in general. In order to select a physically meaningful distributional solution, motivated from the Second Law of thermodynamics (cf. [35, Section 4.5]) the notion of entropy solutions has been introduced in [72, 75], with their analogs in terms of mild solutions [23], kinetic solutions [81, 93] and dissipative solutions [94]. In the case of a homogeneous flux, that is $F(t,x,u) = F(u)$, roughly speaking, an entropy solution to (2.3) is a function u such that for all convex functions $\eta \in C^1(\mathbb{R})$, in the sense of distributions,

$$\mathrm{div}_{t,x}\begin{pmatrix}\eta(u)\\ \eta^F(u)\end{pmatrix} := \partial_t\eta(u) + \mathrm{div}_x\,\eta^F(u) \le 0,$$

where η^F is such that $(\eta^F)'(u) = F'(u)\eta'(u)$. In other words, for entropy solutions entropy-entropy flux pairs (η,η^F) are dissipated at a non-positive rate. Entropy solutions to (2.3) appear as limits of vanishing viscosity approximations

$$\partial_t u^\varepsilon + \mathrm{div} F(t,x,u^\varepsilon) = \varepsilon\Delta u^\varepsilon, \quad \text{on } [0,T]\times\mathbb{R}^d.$$

Noise may enter (nonlinear) SCL in several ways. A statistical analysis of Burgers' equation with white noise as initial condition was given in [8, 17, 100, 102]. The first approach to (inviscid) Burgers equations with random forcing appeared in [90] where the stochastic Burgers equation driven by additive noise was considered. In the ground-breaking work [44] the existence and uniqueness of an invariant measure for the stochastic Burgers equation driven by additive Wiener noise

$$du + \frac{1}{2}\partial_x u^2 = dW_t, \tag{2.4}$$

has been shown. The existence of such an invariant measure is not obvious, since the dissipation mechanism needed in order to compensate the addition of energy via the stochastic term is not readily apparent in (2.4). Roughly speaking, the dissipation of energy is due to the loss of energy in the shocks. Generalizations to more general fluxes, boundary value problems and fractional Brownian motion may be found in [40, 70, 101, 105].

Simultaneously, significant progress has been made in the case of scalar conservation laws perturbed by multiplicative noise. In [67] Lipschitz multiplicative perturbations to general scalar conservation laws with smooth flux F, i.e.

$$du + \partial_x F(u)dt = h(t,x,u)dt + g(u)dW_t \tag{2.5}$$

have been considered via an operator splitting approach. Feng and Nualart extended the theory of multidimensional stochastic SCL in [48] to multiplicative noise by adapting the (deterministic) notion of entropy solutions to the stochastic case, which led to the notion of *strong entropy solutions*. While this notion allowed an immediate adaptation of the classical Kruzkov uniqueness method to the stochastic case, the existence of strong entropy solutions could be proven only in one spatial dimension via the compensated compactness method. This obstacle was (partially) resolved in [39, 66] by adapting the notion of kinetic solutions to stochastic SCL at least for fluxes not depending on time nor on space ($F(t,x,u) \equiv F(u)$). An alternative proof of existence of strong entropy solutions was given more recently in [22] by proving uniform spatial and temporal regularity properties for the vanishing viscosity approximations in $L^1(\mathbb{R}^d)$. Notably, the methods employed in [22] may also be used for time and space dependent fluxes. Finally, the problem of existence of strong entropy solutions was resolved in [11] since the authors managed to modify Kruzkov's uniqueness technique in such a way that uniqueness could be shown for a weaker notion of entropy solution (whose existence is simple). There are several extensions available [12–14]. It remains to mention [71], where well-posedness for (linear) transport equations with multiplicative noise has been shown.

All of the above mentioned works consider *semilinear* stochastic scalar conservation laws in the sense that the noise coefficients do not depend on the derivative(s) of the solution. In contrast, in the recent works [60, 78–80] stochastic perturbations of the flux F are considered, which in general lead to SPDE of the type

$$du_t + \sum_{k=1}^{m} \partial_k F_k(t,x,u_t) \circ d\beta_t^k = 0. \tag{2.6}$$

One motivation of SPDE of this type, besides their occurrence in mean field games (cf. [78]), comes from the relation to stochastic Hamilton-Jacobi-Bellman (HJB) equations. More precisely, if we consider the stochastic HJB equation

$$dv_t + F(t,x,\partial_x v_t) + H(t,x,\partial_x v_t) \circ d\beta_t = 0$$

and set $u_t = \partial_x v_t$ then u_t informally satisfies

$$du_t + \partial_x F(t,x,u_t) + \partial_x H(t,x,u_t) \circ d\beta_t = 0.$$

In the case of homogeneous flux (i.e. $F(t,x,u) \equiv F(u)$) solutions to (2.6) can easily be defined by first passing to the corresponding kinetic formulation. Since this turns (2.6) into a (linear) kinetic equation, one may then define solutions via the corresponding (random) flow of characteristics. This approach was extended based on rough path estimates in [60], in order to treat spatially inhomogeneous fluxes, as they appear in (2.6).

2.2. **Well-posedness by noise for stochastic inhomogeneous scalar conservation laws.** In Section 1.2 we have recalled that stochastic perturbations may lead to well-posedness in the case of transport equations with irregular drift. While part of the motivation of these results was the (informal) relation to questions of uniqueness of solutions to nonlinear PDE arising in fluid mechanics, the methods recalled in Section 1.2 highly depend on the linear structure of the transport equation. In addition, the example (1.22) seems to indicate that an analogous well-posedness by noise effect is not expected for nonlinear PDE. Accordingly, it was concluded in [52]: *The generalization to nonlinear transport equations, where b depends on u itself, would be a major next step for applications to fluid dynamics but it turns out to be a difficult problem* (cf. [52, p.6, l.11 ff.]).

In contrast to this, in the recent work [59] it was shown that a similar effect of well-posedness by noise holds for inhomogeneous scalar conservation laws. More precisely, as a special case, in [59] inhomogeneous Burgers equations of the type

$$\partial_t u + b(x) \cdot \nabla u^2 = 0 \quad \text{on } \mathbb{R}^d, \tag{2.7}$$

were considered. Distributional solutions to (2.7) are, in general, non-unique even if $d = 1$, $b \equiv 1$. However, in this case uniqueness can be restored by restricting to so-called entropy solutions. In the spatially inhomogeneous case (2.7) this ceases to be the case if b is not regular enough. For example, we may consider the model example

$$b(x) = \operatorname{sgn}(x)(\sqrt{|x|} \wedge K) \tag{2.8}$$

for some $K > 0$, $u_0(\cdot) = 1_{[0,1]}(\cdot)$, $d = 1$. For any given time $T > 0$ and choosing $K > T + 1$ for simplicity, we have at least two entropy solutions to (2.7) given by

$$u^1(t,x) := \begin{cases} 1 & \text{if } 0 \le x \le (t+1)^2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$u^2(t,x) := \begin{cases} 1 & \text{if } -t^2 \le x \le (t+1)^2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, entropy solutions to (2.7) are not necessarily unique.

The study of scalar conservation laws with irregular flux has attracted much interest in recent years, see [3–5, 24, 25] among many more, and several selection criteria among entropy solutions have been introduced. One should note, however, that typically divergence form equations with discontinuous fluxes are studied in these works.

In [59] it is shown that for (2.7) noise can have a regularizing effect, in the sense that entropy solutions to

$$du + b(x) \cdot \nabla u^2\, dt + \nabla u \circ d\beta_t = 0 \quad \text{on } \mathbb{R}^d, \tag{2.9}$$

are unique if we assume that $b \in (L^\infty \cap W^{1,1}_{loc})(\mathbb{R}^d)$ and $\operatorname{div} b \in (L^1 \cap L^p)(\mathbb{R}^d)$ for some $p > d$. In particular, these assumptions are satisfied by the model example (2.8). Hence, a well-posedness by noise effect rather similar to the one observed in the linear case in [52] also appears in the case of nonlinear conservation laws. The regularizing effect of the noise appears with regard to irregularities of the flux function rather than with regard to irregularities of the solution caused by the nonlinear nature of the PDE. In fact, also in the stochastically perturbed case shocks still appear and distributional solutions to (2.9) will still be non-unique.

The following result is a special case of the main result obtained in [59].

Theorem 1. *Assume that $b \in (L^\infty \cap W^{1,1}_{loc})(\mathbb{R}^d)$ and that $\operatorname{div} b \in (L^1 \cap L^p)(\mathbb{R}^d)$ for some $p > d$, $p \le \infty$. For every initial condition $u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$ there exists a unique entropy solution u to* (2.9).

The proof of this result relies on passing to a kinetic form of (2.9). In the deterministic case this goes back to [81]. Given a function $u \in L^2([0,T] \times \mathbb{R}^d)$ we introduce the kinetic function

$$\chi(t,x,\xi) = \chi(u(t,x),\xi) := \begin{cases} 1 & \text{for } 0 < \xi < u(t,x) \\ -1 & \text{for } u(t,x) < \xi < 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

Informally, u is an entropy solution to (2.9) if and only if χ solves

$$\partial_t \chi + \xi b(x) \cdot \nabla \chi + \nabla \chi \circ d\beta_t = \partial_\xi m, \tag{2.11}$$

where m is a nonnegative random measure on $[0,T] \times \mathbb{R}^d \times \mathbb{R}$. An advantage of the kinetic form (2.11) is that, in contrast to (2.9), the kinetic equation (2.11) is a linear equation in χ.

The proof of the existence of entropy solutions to (2.9) relies on considering smooth approximations $(b^\varepsilon, u^\varepsilon)$ of (b, u_0). The existence and uniqueness of an entropy solution to

$$du^\varepsilon(t,x) + b^\varepsilon(x) \cdot \nabla (u^\varepsilon)^2(t,x)\, dt + \nabla u^\varepsilon(t,x) \circ d\beta_t = 0$$
$$u^\varepsilon(0,x) = u_0^\varepsilon(x)$$

can then be shown by a simple transformation method. In addition, one may obtain uniform bounds of the type, for all $p \in [1,\infty)$,

$$\operatorname{esssup}_{\omega\in\Omega} \sup_{t\in[0,T]} \|u^\varepsilon(t)\|_{L^p(\mathbb{R}^d)} \lesssim \|u_0\|_{L^p(\mathbb{R}^d)}^p + \|u_0\|_{L^\infty(\mathbb{R}^d)}^{p+1} \|\operatorname{div} b\|_{L^1([0,T]\times\mathbb{R}^d)}. \tag{2.12}$$

This allows to extract weakly and weakly* convergent subsequences $u^{\varepsilon_n} \rightharpoonup u$, $m^{\varepsilon_n} \rightharpoonup m$ as well as $\chi^{\varepsilon_n} = \chi(u^{\varepsilon_n}) \rightharpoonup f$ for some $f \in (L^1_{t,x,\xi} \cap L^\infty_{t,x,\xi})(\mathbb{R}^d)$. The difficulty now is that the limits u and f are not known to satisfy the nonlinear relation (2.10) anymore. This naturally leads to the definition of a generalized entropy solution to (2.11), which relies on weakening the nonlinear relation (2.10) which thereby becomes stable under weak limits. Roughly speaking, a function f is defined to be a generalized entropy solution to (2.9) if f solves (2.11) for some nonnegative measure m and

$$|f| = \operatorname{sgn}(\xi) f \le 1,\ \partial_\xi f = 2\delta_0 - \nu \tag{2.13}$$

for some nonnegative measure ν. The reader may easily verify that entropy solutions are generalized entropy solutions. In view of (2.12) the existence of generalized entropy solutions to (2.11) is obtained in [59] under the weaker assumptions $b \in L^1_{loc}(\mathbb{R}^d)$, $\operatorname{div} b \in L^1(\mathbb{R}^d)$ and $u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$. One should note that for the proof of the existence of a generalized entropy solution, the presence of the stochastic perturbation in (2.11) was not used. The main difficulty thus becomes the proof that generalized entropy solutions to (2.12) are in fact entropy solutions and that entropy solutions are unique.

In order to show that a generalized entropy solution f is an entropy solution, one needs to show that there is a function u such that $f = \chi(u)$. In view of (2.13) it is enough to prove $|f| \in \{0,1\}$ a.e.. To do so, in [59] the difference $|f| - f^2$ is estimated using (2.11). In a first step, one realizes that, since $b \in W^{1,1}_{loc}(\mathbb{R}^d)$, weak solutions to (2.11) are renormalized. Informally, this means that $|f| - f^2$ satisfies

$$\partial_t(|f| - f^2) + \xi b(x) \cdot \nabla(|f| - f^2) + \nabla(|f| - f^2) \circ d\beta_t = (\operatorname{sgn}(\xi) - 2f)\partial_\xi m.$$

Passing to the Itô form, integrating in ξ, taking expectation and using $\partial_\xi f = 2\delta_0 - \nu \le 2\delta_0$ this implies

$$\int \varphi_t \mathbb{E}(|f_t| - f_t^2) \le \int \varphi_0 \mathbb{E}(|f_0| - f_0^2) + \int_0^t \int \mathbb{E}(|f| - f^2)(\partial_t \varphi + \xi \operatorname{div}(b(x)\varphi) + \Delta\varphi)\, dx\, d\xi\, dr$$

for each nonnegative test function $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ not depending on the ξ-variable. Consequently, we would like to solve the parabolic, linear backward PDE

$$\partial_t \varphi + \operatorname{div}(\xi b(x)\varphi) + \Delta\varphi \le C,$$

for some constant $C > 0$ and then use Gronwall's inequality. The difficulty here is that the test-function φ should not depend on ξ, despite the ξ-dependence of the PDE. The main idea is to split up the PDE in two parts, that is,

$$\begin{aligned}\partial_t \varphi + \operatorname{div}(\xi b(x)\varphi) + \Delta\varphi &= \partial_t\varphi + \xi(\operatorname{div} b)\,\varphi + \xi b(x)\cdot\nabla\varphi + \Delta\varphi\\ &\le \partial_t\varphi + R|\operatorname{div} b|\,\varphi + R\|b\|_{L^\infty}\|\nabla\varphi\|_{L^\infty} + \Delta\varphi,\end{aligned}$$

where we used that, since $\|u_0\|_\infty < \infty$, we can restrict to $|\xi| \le \|u_0\|_\infty =: R$. Hence, choosing φ to be a solution to

$$\partial_t\varphi + R|\operatorname{div} b|\,\varphi + \Delta\varphi = 0, \quad \varphi_t = \psi,$$

we obtain that

$$\int \varphi_t \mathbb{E}(|f_t| - f_t^2) \le \int \varphi_0 \mathbb{E}(|f_0| - f_0^2) + \int \mathbb{E}(|f| - f^2)(R\|b\|_{L^\infty(\mathbb{R}^d)}\|\nabla\varphi\|_{L^\infty(\mathbb{R}^d)})dt dx d\xi.$$

It then only remains to prove that $\|\varphi\|_{L^\infty(\mathbb{R}^d)} + \|\nabla\varphi\|_{L^\infty(\mathbb{R}^d)}$ is finite, which follows from heat kernel estimates.

2.3. **Regularization by noise for stochastic scalar conservation laws.** In Section 2.2 we have seen that linear multiplicative transport noise can lead to well-posedness of entropy solutions in the case of (nonlinear) scalar conservation laws with spatially irregular flux in a similar fashion as it does in the linear case. In other words, also in the nonlinear situation spatial inhomogeneities can be regularized by this type of noise. On the other hand, the example (1.22) shows that singularities that are due to the nonlinear structure of the equation, e.g. shocks, are not affected by linear transport noise. The aim of the following two sections is to present results obtained in [58, 61, 79] where it is shown that, in contrast, a certain type of *nonlinear* noise can be used to regularize singularities caused by nonlinear effects in some PDE.

The example (1.22) shows that linear multiplicative noise does not lead to higher regularity for solutions to Burgers' equation. We recall that the kinetic form of the Burgers equation

$$\partial_t u + \partial_x u^2 = 0 \quad \text{on } [0,T]\times\mathbb{T} \tag{2.14}$$

introduced in the last section, reads

$$\partial_t \chi + \xi\partial_x\chi = \partial_\xi m, \tag{2.15}$$

for some nonnegative measure m, were $\mathbb{T}$ denotes the one-dimensional torus. The nonnegativity of the entropy defect measure m corresponds to the restriction to entropy solutions to (2.14). When we drop this restriction, that is, we consider weak solutions to (2.14) such that the kinetic function $\chi = \chi(u)$ satisfies (2.15) for some finite Radon measure m, not necessarily nonnegative, we are let to the class of quasi-solutions to (2.14). One reason to work with this class of solutions is that regularity estimates obtained from averaging techniques are essentially sharp for quasi-solutions. More precisely, following the arguments of [63, 68] one can see that quasi-solutions to (2.14) satisfy $u(t) \in W^{\lambda,1}$ for every $\lambda \in (0, \frac{1}{3})$. As shown in [38], however, there are quasi-solutions u such that $u(t) \notin W^{\lambda,1}$ for every $\lambda > \frac{1}{3}$.

In [61], among other results, the regularity of solutions to scalar conservation laws, including as a special case the stochastic Burgers equation

$$du + \partial_x u^2 \circ d\beta_t = 0 \ \text{ in } \ \mathbb{T}\times(0,\infty), \tag{2.16}$$

has been considered and it was shown that quasi-solutions to (2.16) satisfy $u(t) \in W^{\lambda,1}$ for every $\lambda \in (0, \frac{1}{2})$. In this sense, we see that the noise included in (2.16) has a regularizing effect. More precisely, the corresponding result in [61] reads

Theorem 2. *Let u be a pathwise quasi-solution to* (2.16) *with $u_0 \in L^2(\mathbb{T})$. Then, for all $\lambda \in (0, \frac{1}{2})$ and $T > 0$, there is a $C > 0$ such that*

$$\mathbb{E}\int_0^T \|u(t)\|_{W^{\lambda,1}} dt \le C(1 + \|u_0\|_2^2 + \mathbb{E}|m|([0,T] \times \mathbb{T} \times \mathbb{R}))$$

and, for all $\delta > 0$,

$$\sup_{t \ge \delta} \mathbb{E}\|u(t)\|_{W^{\lambda,1}} < \infty.$$

While this shows an improvement of regularity of quasi-solutions by the inclusion of noise, one should note that if $u_0 \in BV(\mathbb{T})$ then each entropy solution to (2.14) satisfies $u(t) \in BV(\mathbb{T})$ for all $t \ge 0$. Hence, in this case no regularizing effect becomes apparent by the methods of [61]. This obstacle will be addressed in the following section.

2.4. **Open interfaces and porous media equations.** In order to outline the developments of [58] we shall concentrate on the model case of the porous medium equation

$$\partial_t w = \frac{1}{12}\partial_{xx} w^3 \quad \text{on } \mathbb{R}, \tag{2.17}$$

with initial condition w_0 being non-negative, smooth and compactly supported. Informally, one may rewrite this equation as

$$\partial_t w = \frac{1}{4} w^2 \partial_{xx} w + \frac{1}{2} w |\partial_x w|^2.$$

Concentrating at the leading order term $w^2 \partial_{xx} w$ we see that the diffusivity coefficient w^2 is large for large values of w but decays to zero for $w \downarrow 0$. This leads to mass building up at the open interface

$$I(t) := \partial \operatorname{supp} w(t, \cdot)$$

and causes the possibility of singularities in terms of a blow-up of $\|\partial_x w(t)\|_{L^\infty}$ even if w_0 is smooth. Indeed, in the long-run solutions converge to the so-called Barenblatt solutions (cf. [106]) given by

$$w(t,x) = t^{-\frac{1}{4}}(C - \frac{1}{12}|x|^2 t^{-\frac{1}{2}})_+^{\frac{1}{2}},$$

where C depends on the L^1 norm of the initial condition. Clearly, at the open interface

$$I(t) = \{x \in \mathbb{R} : C - \frac{1}{12}|x|^2 t^{-\frac{1}{2}} = 0\} = \{\pm\sqrt{12C} t^{\frac{1}{4}}\},$$

the (informal) derivative

$$\partial_x w(t,x) = -\frac{1}{12} t^{-\frac{3}{4}} (C - \frac{1}{12}|x|^2 t^{-\frac{1}{2}})_+^{-\frac{1}{2}} 1_{C \ge \frac{1}{12}|x|^2 t^{-\frac{1}{2}}} x,$$

is unbounded. In [58] the possibility of regularizing this singularity by perturbation with nonlinear noise was investigated. The principle idea put forward in [58] is to regularize the singularity observed for (2.17) by the inclusion of noise of the type (2.16), that is, to consider, for $\sigma > 0$,

$$dv + \frac{\sigma}{2}\partial_x v^2 \circ d\beta_t = \frac{1}{12}\partial_{xx} v^3 \, dt \quad \text{on } \mathbb{R}. \tag{2.18}$$

In [62], the well-posedness and regularity of solutions to (2.18) was shown. As in [61] it remained an open question if this regularity is optimal. More importantly, the regularity estimates for solutions to (2.18) proven in [62] do not improve the regularity known in the deterministic case (2.17). These questions are addressed in [58].

Indeed, the results obtained in [58] prove that the solution v to (2.18) satisfies, if $\sigma > 1$, for all $t \geq 0$,

$$v(t) \in W^{1,\infty}(\mathbb{R}) \quad \mathbb{P}\text{-a.s.},$$

whereas (at least for some choice of initial conditions), if $\sigma \leq 1$, one has

$$\mathbb{P}\text{-a.s.} \quad \exists T > 0, \forall t \geq T, v(t) \notin W^{1,\infty}(\mathbb{R}).$$

More precisely, in [58] the following *sharp* bound is shown:

$$\|\partial_x v(t)\|_{L^\infty} \leq \frac{1}{L^+(t) \wedge L^-(t)}, \tag{2.19}$$

where L^+, L^- are the maximal continuous solutions to the reflected SDE on $(0, \infty)$ given by

$$\begin{aligned} dL^+ &= -\frac{1}{2L^+(t)}dt + \sigma d\beta_t, \quad L^+(0) = \frac{1}{\|(\partial_x v_0)_+\|_{L^\infty}} \\ dL^- &= -\frac{1}{2L^-(t)}dt - \sigma d\beta_t, \quad L^-(0) = \frac{1}{\|(\partial_x v_0)_-\|_{L^\infty}}. \end{aligned}$$

This demonstrates that, when the noise coefficient is large enough, the stochastic perturbation in (2.18) has a regularizing effect as compared to the non-perturbed situation (2.17) for which $\|\partial_x w\|_{L^\infty}$ may blow up in finite time. An interesting point about this result is that the observed regularizing effect depends on the strength of the noise σ, in contrast to the linear case (1.18). Moreover, it is shown in [58] that the estimate in (2.19) is optimal, in the sense that for a class of initial conditions equality in (2.19) holds.

We shall close the account of the results obtained in [58] by stating the main result in its general form. Consider SPDE of the type

$$du + \frac{1}{2}|Du|^2 \circ d\xi_t = F(x, u, Du, D^2u)\, dt \quad \text{on } \mathbb{R}^d, \tag{2.20}$$

for $F : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$ and $\xi : \mathbb{R}_+ \to \mathbb{R}$ being continuous functions. The map F is supposed to satisfy the typical conditions from viscosity theory allowing to prove the existence and uniqueness of viscosity solutions to (2.20) (cf. e.g. [57, 82]). In addition, and more importantly for our sake, F is assumed to satisfy the following: there exists a locally Lipschitz continuous function $V_F : (0, \infty) \to \mathbb{R}$, bounded from above on $[1, \infty)$ such that for all $g \in BUC(\mathbb{R}^d)$, $t \geq 0$, one has, in the sense of distributions,

$$D^2 g \leq \ell_0^{-1} Id \quad \Rightarrow \quad D^2(S_F(t, g)) \leq \frac{Id}{\varphi^{V_F}(t)(\ell_0)},$$

where $\varphi^{V_F}(t) : \mathbb{R}_+ \to \bar{\mathbb{R}}_+$ is the flow of solutions to the ODE

$$\dot{\ell}(t) = V(\ell), \quad \ell(0) = \frac{1}{\ell_0},$$

stopped when reaching the boundaries 0 or $+\infty$ and $S_F(t, g)$ is the viscosity solution to the deterministic PDE

$$du = F(x, u, Du, D^2u)\, dt \quad \text{on } \mathbb{R}^d.$$

This assumption yields a control on the rate of loss of semiconcavity for S_F. Note that φ^{V_F} may take the value 0 and thus no preservation of semiconcavity is assumed. In the following $BUC(\mathbb{R}^d) = UC_b(\mathbb{R}^d)$ will denote the space of bounded, uniformly continuous functions on $\mathbb{R}^d$.

Theorem 3. *Let $u_0 \in BUC(\mathbb{R}^d)$, $\xi \in C(\mathbb{R}_+)$, assume that F satisfies the above assumptions and let u be the solution to*

$$\begin{cases} du + \frac{1}{2}|Du|^2 \circ d\xi(t) = F(t,x,u,Du,D^2u)dt, \\ u(0,\cdot) = u_0. \end{cases} \tag{2.21}$$

Suppose that $D^2 u_0 \leq \frac{Id}{\ell_0}$ for some $\ell_0 \in [0,\infty)$, in the sense of distributions. Then, for each $t \geq 0$,

$$D^2 u(t,\cdot) \leq \frac{Id}{L(t)}, \tag{2.22}$$

in the sense of distributions, where L is the maximal continuous solution on $[0,\infty)$ to

$$\begin{aligned} &dL(t) = V_F(L(t))dt + d\xi(t) \ \text{ on } \{t \geq 0 : L(t) > 0\}, \quad L \geq 0, \\ &L(0) = \ell_0. \end{aligned} \tag{2.23}$$

The proof is based on a Trotter-Kato splitting scheme for (2.20). The estimate (2.22) is then proven for the corresponding approximating solutions u^n with respect to a discretization L^n of L, based on semiconcavity estimates for S_H, with $H(p) = \frac{1}{2}|p|^2$, where S_H denotes the solution to

$$\partial_t u + H(Du) = 0.$$

It is well-known that S_H and S_{-H} allow to obtain one-sided bounds (of the opposite sign) on the second derivative (cf e.g. [77]), and the fact that one can combine these two bounds to obtain $C^{1,1}$ bounds goes back to Lasry and Lions [74].

More precisely, setting $t_i^n = \frac{ti}{n}$ the approximations

$$u^n(t) := S_H(\xi_{t_{n-1}^n, t_n^n}) \circ S_F(\frac{t}{n}) \circ \cdots \circ S_H(\xi_{t_0^n, t_1^n}) \circ S_F(\frac{t}{n}) u^0$$

are considered, where $\xi_{s,t} := \xi_t - \xi_s$. By an extension of the continuity property of solutions to (2.20) with respect to the continuous driving path ξ to piecewise constant paths, it is shown in [58] that

$$u(t,\cdot) = \lim_{n\to\infty} u^n(t,\cdot).$$

Combining the above mentioned semiconcavity estimates for S_H with the assumption on S_F yields the following discrete analog of the bound (2.22)

$$D^2 u^n(t,\cdot) \leq \frac{Id}{L^n(t)},$$

where L^n is defined by induction

$$L^n(0) = \ell_0, \quad L^n(t_i^n) = \left(\varphi^{V_F}(\frac{t}{n})(L^n(t_{i-1}^n)) - \xi_{t_{i+1}^n, t_i^n}\right)_+.$$

The remaining difficulty is to prove the convergence of L^n to L, treated in detail in [58].

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