

Synchronization of long-time asymptotics by noise.

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Outline

- 1 The deterministic case
- 2 The stochastic case
 - Non-degenerate SPDE
 - Degenerate SPDE with additive noise
 - Degenerate SPDE with multiplicative noise

The deterministic case

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The deterministic case

- Non-degenerate PDE (Chafee-Infante equation)

$$dX_t = \Delta X_t dt + (\lambda X_t - X_t^3) dt, \quad \lambda \geq 0. \quad (\text{RDE})$$

vs. degenerate PDE (porous medium equation, $m > 1$)

$$dX_t = \Delta X_t^{[m]} dt + \lambda X_t dt, \quad \lambda \geq 0. \quad (\text{PME})$$

For simplicity write: $X^{[m]}$ for $|X|^{m-1}X$.

- Both with zero Dirichlet boundary conditions on bounded domains $\mathcal{O} \subseteq \mathbb{R}^d$.

Reaction diffusion equations

- Linearization of Chafee-Infante:

$$dX_t = \Delta X_t dt + \lambda X_t dt$$

- Expansion of X_t in eigenvectors e_k of $-\Delta$: $X_t = \sum_k X_t^k e_k$ yields

$$dX_t^k = (\lambda - \lambda_k) X_t^k dt$$

- For $\lambda = 0$ the invariant solution $X \equiv 0$ is (exponentially) stable.
- As λ increases 0 bifurcates whenever λ crosses eigenvalues λ_k .
- Consequence: Finite dimensional attractor with explicitly known dimension (for Chafee-Infante).

Porous medium equations

- PME:

$$\begin{aligned} dX_t &= \Delta X_t^{[m]} dt + \lambda X_t dt \\ &= X_t^{m-1} \Delta X_t dt + X_t^{[m-2]} |\nabla X_t|^2 dt + \lambda X_t dt. \end{aligned} \quad (\text{PME})$$

- Intuitively: Ellipticity constant at the invariant solution $X \equiv 0$ degenerates \rightarrow all modes become unstable.
- Technical obstacle: Cannot proceed by linearization. How to get lower bounds?
- [Efendiev, Zelik; 2008]: Attractor of (PME) has infinite fractal dimension for $\lambda > 0$.

The stochastic case

The stochastic case

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Synchronization by noise?	non-degenerate ellipticity	degenerate ellipticity
additive noise		
multiplicative noise		

Stochastic reaction diffusion equation

Case 1: Additive noise

- Consider

$$dX_t = \Delta X_t dt + \lambda X_t dt + dW_t.$$

- Assume $W_t = \sum_{k=1}^{\infty} \sigma_k e_k \beta_t^k$.
- Expansion in eigenmodes: $X_t = \sum_k X_t^k e_k$ yields

$$dX_t^k = (\lambda - \lambda_k) X_t^k dt + \sigma_k d\beta_t^k.$$

- Ergodicity for Chafee-Infante if $\sigma_k \neq 0$ for all $\lambda > \lambda_k \rightarrow$ synchronization by noise on the level of ergodicity

$$\mathcal{L}(X_t) \rightarrow_{TV} \mu.$$

Stochastic reaction diffusion equation

Synchronization

- Reminder of [Caraballo, Crauel, Langa, Robinson; 2007], related [Chueshov, Scheutzow; 2004].
- RDS given by $\varphi(t, \omega)x = X_t^x(\omega)$.
- Known: X ergodic, φ has a random attractor \mathcal{A} .
- \mathcal{A} is bounded in L^∞
 \rightarrow can choose $\underline{a}(\omega) = \inf \mathcal{A}(\omega)$, $\bar{a}(\omega) = \sup \mathcal{A}(\omega)$:

$$\underline{a}(\omega) \leq \mathcal{A}(\omega) \leq \bar{a}(\omega).$$

- φ is order preserving, i.e. $x \leq y$ then $\varphi(t, \omega)x \leq \varphi(t, \omega)y$
- Then \underline{a}, \bar{a} are invariant:

$$\varphi(t, \omega)\bar{a}(\omega) = \bar{a}(\theta_t \omega)$$

$$\varphi(t, \omega)\underline{a}(\omega) = \underline{a}(\theta_t \omega)$$

- Hence, $\mathcal{L}(\underline{a}), \mathcal{L}(\bar{a})$ are invariant measures for X_t . By ergodicity $\mathcal{L}(\underline{a}) = \mathcal{L}(\bar{a})$, thus $\underline{a} = \bar{a}$ and

$$\mathcal{A}(\omega) = \{\bar{a}(\omega)\}.$$

Stochastic reaction diffusion equation

Synchronization by noise?	non-degenerate ellipticity	degenerate ellipticity
additive noise	✓	
multiplicative noise		

Stochastic reaction diffusion equation

Multiplicative Stratonovich noise

- Spatially homogeneous noise:

$$dX_t = \Delta X_t dt + \lambda X_t dt + \sigma X_t \circ d\beta_t$$

Transformation: $Y_t = e^{-\sigma\beta_t} X_t$ solves

$$dY_t = \Delta Y_t dt + \lambda Y_t dt$$

→ *No* regularization.

- Spatially inhomogeneous noise: [Caraballo, Robinson; 2004]

$$dX_t = \Delta X_t dt + \lambda X_t dt + \sum_{k=1}^{\infty} B_k X_t \circ d\beta_t^k,$$

B_k suitable linear operators. It is enough to have B_k acting on the finite-dimensional unstable part. Under proper assumptions on B_k system becomes ergodic.

Stochastic reaction diffusion equation

Synchronization by noise?	non-degenerate ellipticity	degenerate ellipticity
additive noise	✓	
multiplicative noise	✓	

Degenerate SPDE with additive noise

Degenerate SPDE with additive noise based on [G., JDDE, 2013]

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Degenerate SPDE with additive noise

- Consider

$$dX_t = \Delta X_t^{[m]} dt + \lambda X_t dt + dW_t, \quad m > 1$$

with zero Dirichlet boundary conditions on a bounded domain.

- Aim: Prove synchronization.
- Tasks
 - Construct stochastic flow
 - Prove existence of a random attractor \mathcal{A}
 - Show \mathcal{A} is 0-dimensional (Note: Stabilization of infinitely many unstable modes).
- Obstacles:
 - Construction of stochastic flow requires regular noise W_t [Beyn, G., Lescot, Röckner; CPDE, 2011]
 - Ergodicity requires non-degenerate noise [Liu; JEE, 2009]
- Solved in [G., JDDE, 2013]

Degenerate SPDE with additive noise

- RDS to

$$dX_t = \Delta X_t^{[m]} dt + \lambda X_t dt + dW_t \quad (\text{SPME})$$

usually is constructed via transformation $Y_t = X_t - W_t$ yields

$$dY_t = \Delta(Y_t + W_t)^{[m]} dt + \lambda(Y_t + W_t) dt.$$

Solve pathwise \rightarrow RDS.

- Problem: Only makes sense if W_t takes values in $\mathcal{D}(\Delta(\cdot)^{[m]})$.
- Solution: Construct strictly stationary solution Z_t (\sim stationary, nonlinear Ornstein-Uhlenbeck process) to strongly monotone part of (SPME), i.e. to

$$dX_t = \Delta X_t^{[m]} dt + dW_t.$$

Exists by dissipativity method.

- Transform (SPME) by $Y_t = X_t - Z_t$ into a random PDE. Advantage: Z_t takes values in $\mathcal{D}(\Delta(\cdot)^{[m]})$.
- Result: Existence of RDS for trace-class noise W_t .

Degenerate SPDE with additive noise

Theorem

Assume $d \leq 2$. Then there is a minimal weak point attractor for

$$dX_t = \Delta X_t^{[m]} dt + \lambda X_t dt + dW_t \quad (\text{SPME})$$

consisting of a single random point.

Synchronization by noise?	non-degenerate ellipticity	degenerate ellipticity
additive noise	✓	✓
multiplicative noise	✓	

Degenerate SPDE with multiplicative noise

Degenerate SPDE with multiplicative noise based on [G., SIMA, 2013]

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Degenerate SPDE with multiplicative noise

- Recall: Reaction-Diffusion equations may be stabilized by space-time Stratonovich noise.

- Consider:

$$dX_t = \Delta X_t^{[m]} dt + \lambda X_t dt + \sum_{k=1}^N f_k X_t \circ d\beta_t^k. \quad (\text{SPME})$$

- Recall that in the deterministic case ($f_k \equiv 0$, $\lambda > 0$) the attractor has infinite fractal dimension.
- Question: May we still stabilize (SPME) by multiplicative space-time Stratonovich noise?
- Aim: We show that the random attractor for (SPME) has infinite fractal dimension, i.e. no stabilization occurs.

Construction of an RDS

- Recall:

$$dX_t = \Delta X_t^{[m]} dt + \lambda X_t dt + \sum_{k=1}^N f_k X_t \circ d\beta_t^{(k)}.$$

- Set $Y_t := e^{\mu t} X_t$, where $\mu_t = -\lambda t - \sum_{k=1}^N f_k \beta_t^{(k)}$. Then

$$\partial_t Y(t, x) = e^{\mu(t, x)} \Delta \left(e^{-\mu(t, x)} Y(t, x) \right)^{[m]}.$$

- Existence and uniqueness in [Barbu, Röckner, JDE, 2011], [G., AoP, 2013].

Degenerate SPDE with multiplicative noise

Idea of the proof:

- The invariant solution 0 is unstable, has ∞ -dimensional unstable manifold.

$$\mathcal{M}^+(0, \omega) := \{u_0 \in X \mid \exists u : (-\infty, 0] \rightarrow X, \text{ such that } \varphi(t; \theta_{-t}\omega)u(-t) = u_0 \\ \text{for all } t \geq 0 \text{ and } \|u(t)\|_{L^\infty(\mathcal{O})} \rightarrow 0 \text{ for } t \rightarrow -\infty\}.$$

(Recall: Cannot proceed via linearization)

- In terms of the transformed equation: Find $v \in L^\infty((-\infty, 0] \times \mathcal{O})$ such that

$$u_0 = Y(0, -t; \omega)v(-t), \quad \forall t \geq 0. \quad (*)$$

Degenerate SPDE with multiplicative noise

- Use time transformation to reduce to finite time interval:
 $G(t) = \log(t) : (0, 1] \mapsto (-\infty, 0]$. Get

$$\partial_t U = e^{\mu G(t) + \eta G(t)} \Delta (e^{-\mu G(t) + \eta G(t)} U_t)^m, \text{ on } [0, 1] \times \mathcal{O}. \quad (**)$$

→ L^∞ Solutions to (**) give solutions to (*), i.e. elements $u_0 = v(0)$ of $\mathcal{M}^+(0, \omega)$.

- Need to know that solutions obtained this way have sufficiently different end-value $v(0)$.
 → prove finite speed of propagation for (**).

Theorem (G., SIMA, 2013)

The Kolmogorov ε -entropy of \mathcal{A} is bounded below by

$$\mathbb{H}_\delta(\mathcal{A}(\omega)) \geq C(\omega) \delta^{\frac{-d(m-1)}{2+d(m-1)}}, \quad \forall \omega \in \Omega,$$

In particular, the fractal dimension $d_f(\mathcal{A}(\omega))$ is infinite for all $\omega \in \Omega$.

Conclusion

Synchronization by noise?	non-degenerate ellipticity	degenerate ellipticity
additive noise	✓	✓
multiplicative noise	✓	✗

Outlook

- All the above results on synchronization depend crucially on the RDS being order-preserving
- E.g. Double-well potential in \mathbb{R}^d :

$$dX_t = (X_t - X_t^3)dt + dW_t.$$

- Then φ is order-preserving only in $d = 1$ and synchronization holds.
- Open question: $d \geq 2$.
- Answer: Synchronization still holds, but completely different proof, based on Lyapunov exponents/local stable manifolds.

Thanks

Thanks!