Fluctuations in non-equilibrium and stochastic PDE

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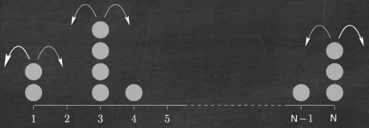
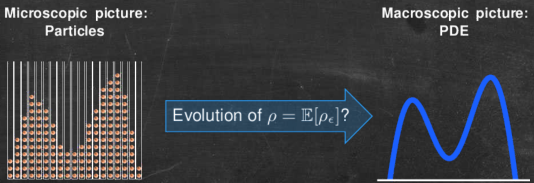
Fluctuations in non-equilibrium and stochastic PDE

- Aim: Fluctuation corrections to continuum models
- Interacting particle system, e.g. exclusion process, zero range process etc.



- Hydrodynamic limit:



- Hydrodynamic limit: Empirical density field $\mu^N(t) \rightharpoonup^* \bar{\rho}(t)\,dx$ with

$$\partial_t \bar{\rho} = \frac{1}{2}\partial_{xx}\Phi(\bar{\rho})$$

with Φ the mean local jump rate.

- Rate of convergence? Central limit fluctuations

$$d(\mu^N, \bar{\rho}\, dx) \approx N^{-\frac{1}{2}}.$$

- Aim 1: Identify fluctuation correction to continuum model, so that a higher order of approximation is reached
- Rare events and large deviations.
- Aim 2: Identify fluctuation correction to continuum model, so that correct large deviations are satisfied.
- Ansatz:

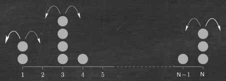
$$\partial_t \rho^N = \partial_{xx}\left(\Phi(\rho^N)\right) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho^N)}dW_t\right),$$

  where dW is space-time white noise.
- Informal justification:
  1. Physics: Fluctuation-dissipation relation, "fluctuating hydrodynamics"
  2. Mean behavior / law of large numbers.
  3. Central limit fluctuations
  4. Large deviations.

- Obstacle: The stochastic PDE

$$\partial_t \rho^N = \partial_{xx}\left(\Phi(\rho^N)\right) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho^N)}dW_t\right)$$

  is not well-posed & supercritical -> no regularity structures
- Decorrelation length of discrete system $= \frac{1}{N}$.



$$\partial_t \rho^N = \partial_{xx}\left(\Phi(\rho^N)\right) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho^N)}dW_t^N\right)$$

  where W^N has correlation length $\frac{1}{N}$.
- Ansatz: joint limit "small noise, ultraviolet cutoff"

$$\partial_t \rho^{N,K} = \partial_{xx}\left(\Phi(\rho^{N,K})\right) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho^{N,K})} \circ dW_t^K\right)$$

  where W^K has correlation length $\frac{1}{K}$.
- Well-posedness: [Lions, Souganidis; 1998ff], [Lions, Perthame, Souganidis; 2013], [Lions, Perthame, Souganidis; 2014], [G., Souganidis; 2014], [G., Souganidis; 2015], [G., Fehrman; 2020], [Dareiotis, G.; 2019], [Fehrman, Gess; 2021].

- Large deviations [Fehrman, Gess; 2020]: Large deviations in the joint scaling limit $N, K \to \infty$. Recovering the correct rate function dictated from the zero range process.

Note: This is a particular case in which the link between Macroscopic fluctuation theory [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015] and fluctuating hydrodynamics [Landau-Lifshitz 1973, Spohn 1991] can be made rigorous.

- Crucial ingredient: Well-posedness for the "skeleton equation"

$$\partial_t \rho = \Delta\Phi(\rho) + \mathrm{div}\left(\sqrt{\Phi(\rho)}g(t,x)\right)$$

with $g \in L^2_{t,x}$.

- Scaling argument: $g \in L^2_{t,x}$ is energy critical.
- Nonlinear generalization of DiPerna-Lions (new nonlinear commutators, optimal regularity, new uniqueness for kinetic solutions)