Synchronization by noise

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Outline

Introduction

Introduction

Synchronization by noise

- We consider SDE on $\mathbb{R}^d$ of the type

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t. \qquad (*)$$

- The inclusion of noise may simplify the long-time dynamics, i.e. while

$$dX_t = b(X_t)dt$$

  may not be globally stable, the long-time behavior of $(*)$ may be trivial.
- Roughly speaking: Synchronization by noise means that the random attractor consists of a single random point, i.e.

$$A(\omega) = \{a(\omega)\}, \quad \mathbb{P}\text{-a.s.}$$

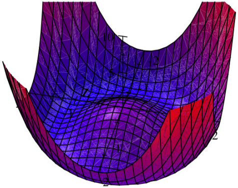
- In particular: If synchronization occurs, then each two trajectories converge to each other in probability:

$$|X_t^x - X_t^y| \to 0 \quad \text{for } t \to \infty$$

  in probability.

Model example

- Double-well potential, $V(x) = -\frac{1}{2}|x|^2 + \frac{1}{4}|x|^4$



with additive Wiener noise, i.e.

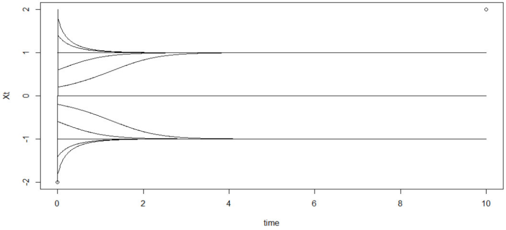
$$dX_t = (X_t - |X_t|^2 X_t)dt + \sigma dW_t.$$

Model example

- Deterministic case ($\sigma = 0,\ d = 1$):

$$dX_t = (X_t - X_t^3)dt$$

- Attractor is given by closed unit ball: $A = \bar{B}_1(0) = [-1, 1]$.
- Point attractor is given by $S^{d-1} \cup \{0\} = \{\pm 1, 0\}$.
- Simulation:

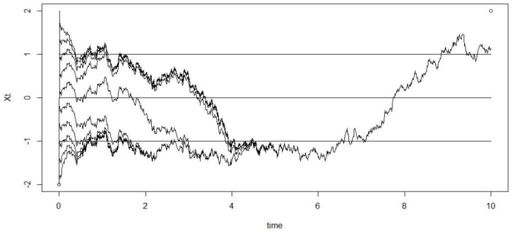


Model example

- Additive noise ($\sigma > 0$):

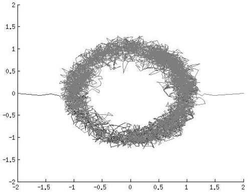
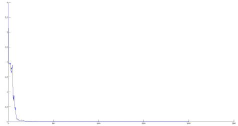
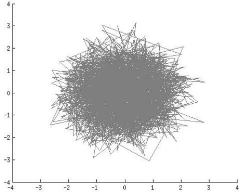
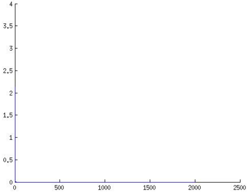
$$dX_t = (X_t - X_t^3)dt + \sigma dW_t$$

- Synchronization occurs: $A(\omega) = \{a(\omega)\}$ a.s.. In particular $|X_t^x - X_t^y| \to 0$ for $t \to \infty$ in probability.
- Simulation:



Model example

- Starting point of presented work: How to prove this for $d > 1$?

	Trajectories	Distance of trajectories
$\sigma = 1$		
$\sigma = 10$		

Background on random dynamical systems

Background on random dynamical systems

Stochastic flows

- Consider
$$dX_t = b(X_t)dt + \sigma(X_t)dW_t. \qquad (*)$$
- By Arnold, Scheutzow, Kunita (among others) one may select a version φ_t of the solution X_t giving a random dynamical system

Definition

Let $\theta := (\theta_t)_{t\in\mathbb{R}}$ be a *metric dynamical system* on $(\Omega, \mathscr{F}, \mathbb{P})$, i.e. θ is a group of measurable and $\mathbb{P}$-preserving maps on Ω.
The map $\varphi : [0,\infty) \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a *random dynamical system* (RDS) if

1. φ is measurable,
2. $x \mapsto \varphi_t(\omega)x$ is continuous for all $t \geq 0$, $\omega \in \Omega$,
3. $\varphi_{t+s}(\omega)x = \varphi_t(\theta_s\omega)\varphi_s(\omega)x, \quad \varphi_0(\omega)x = x$.

Canonical setup: $\Omega = C(\mathbb{R};\mathbb{R}^d)$, $\mathbb{P}$ double-sided Wiener measure, $\theta_t\omega(s) := \omega(t+s) - \omega(t)$, $\varphi_t(\omega)x$ good version of a solution to (*) with i.c. x.

White noise RDS

- For $s \leq t$ we define

$$\mathscr{F}_{s,t} := \sigma(\varphi_{s',t'}, s \leq s' \leq t' \leq t).$$

In particular, let $\mathscr{F}_0 := \mathscr{F}_{-\infty,0}$ be the past of the system up to time 0.

Definition

An RDS φ is a white noise RDS, if $\mathscr{F}_{s,t}$, $\mathscr{F}_{s',t'}$ are independent for all disjoint (s,t), (s',t').

- Let φ be a white noise RDS, then

$$P_t f(x) := \mathbb{E} f(\varphi_t(\cdot)x)$$

defines a Markov semigroup on $C_b(\mathbb{R}^d;\mathbb{R})$.

Random attractors

Definition

A *weak random attractor* is a random set $A(\omega)$ such that

1. (invariance): $\varphi_t(\omega)A(\omega) = A(\theta_t\omega)$, a.s. for all $t \geq 0$.
2. (attraction):
$$d(\varphi_t(\omega)B, A(\theta_t\omega)) \to 0 \quad \text{for } t \to \infty$$
in probability, for each compact set B.
3. (compactness): $A(\omega)$ is compact a.s..

If we replace compact sets B by points, then A is called a *weak point attractor*.

Fact

Weak random attractors are $\mathscr{F}_0$-measurable.

Definition

We say that *synchronization* occurs if the weak random attractor is a singleton

$$A(\omega) = \{a(\omega)\} \quad \text{a.s..}$$

We say that *weak synchronization* occurs if there is a singleton weak point attractor.

Brief overview of known methods

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Known methods

There are several distinct methods to prove synchronization by noise available in the literature (there are many more!):

1. Monotone RDS + uniqueness of invariant measure (e.g. Arnold, Chueshov '98; Chuechov, Scheutzow '04)
2. Local stability + transitivity of the two-point motion (e.g. Baxendale '91)
3. Perturbation techniques/large deviation methods (e.g. Tearne '08, Martinelli, Scoppola, '88, '94)
4. Master-slave synchronization (Chueshov, Schmalfuss '10)
5. ... (many more) ...

Monotone RDS

- Monotone RDS: Assume that there is a partial order $\leq$ of the state space (say $\mathbb{R}^d$) that is preserved by φ, i.e.

$$\text{if } x \leq y \text{ then } \varphi_t(\omega)x \leq \varphi_t(\omega)y.$$

- Assume that there is a random attractor A suitable "compatibility" of $\leq$ with the topoloy on $\mathbb{R}^d$. By Arnold, Chueshov '98 there are random variables $a_-, a_+ \in A$ such that

$$A(\omega) \subseteq [a_-(\omega), a_+(\omega)].$$

- Invariance of A implies that a_-, a_+ are invariant under φ.
- Uniqueness of the invariant measure gives: $\mathscr{L}(a_-) = \mathscr{L}(a_+)$ which implies $a_- = a_+$ a.s..
- Model example: For $d = 1$ this proves synchronization for

$$dX_t = (X_t - X_t^3)dt + \sigma dW_t$$

with $\sigma > 0$.

Local stability + transitivity of the two-point motion

- Assumptions: Local stability + transitivity of the two point motion
- Under suitable ergodic conditions, there is one number λ_{top} , called first (or top) Lyapunov exponent, such that
$$\lambda_{top} = \lim_{t \to \infty} \frac{1}{t} \log |D\varphi_t(x,\omega)v|$$
exists for certain x, v, ω and it is the largest such limit.
- Local stability: $\lambda_{top} < 0$. This yields local stability, e.g. by local stable manifold theorem (e.g. Mohammed, Scheutzow '99)
- How to pass to global stability?
- Baxendale 91': Assume transitivity of the two point motion $t \mapsto (\varphi_t(\omega)x, \varphi_t(\omega)y)$. In particular, i.e. for each $\delta > 0$ the stopping time
$$\tau := \inf\{t \geq 0 | |\varphi_t(\omega)x - \varphi_t(\omega)y| \leq \delta\}$$
is finite with positive probability.
- For additive noise this is not a good assumption
$$d\varphi_t(x) = b(\varphi_t(x))dt + dW_t$$
$$d\varphi_t(y) = b(\varphi_t(y))dt + dW_t.$$
The noise only shifts parallel to the diagonal.

Perturbation techniques

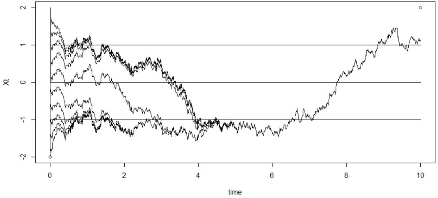
- Tearne '08 for small noise based on dynamical considerations.
- Consider:

$$dX_t = b(X_t)dt + \varepsilon dW_t$$

- Among other assumptions assume:
  - there are finitely many fixed points of b
  - all stable fixed points are hyperbolic
  - the noise is small enough.

Perturbation techniques

- The idea is:
  - each trajectory spends a long time in a basin of attraction, then jumps to another;
  - the trajectories of two different initial conditions may be in two different basins but sooner or later there is a fluctuation that sends both in the same basin and there they approach each other at least for small σ,
  - the time spent in such basins is so long that next fast transitions between basins cannot split again the particles.
- Only covers double well in $1-d$:



- The examples treated by Tearne '08 are also covered by the general theory below.

Model example

Question

Open question in the literature: Does synchronization occur for

$$dX_t = (X_t - |X_t|^2 X_t)dt + \sigma dW_t$$

with $\sigma > 0$ *and* $d > 1$*?*

A new approach to synchronization

A new approach to synchronization

Local stability

In the following let φ be a white noise RDS, (E,d) be a Polish space.

Definition

Let $U \subset E$ be a (deterministic) non-empty open set. We say that φ is *asymptotically stable* on U if there exists a (deterministic) sequence $t_n \uparrow \infty$ such that

$$\mathbb{P}(\lim_{n\to\infty} \operatorname{diam}(\varphi_{t_n}(.,U)) = 0) > 0.$$

- We will see later that a negative top Lyapunov exponent implies asymptotic stability.

Lemma

Let φ be asymptotically stable on U and assume

$$\mathbb{P}(A \subset U) > 0.$$

Then A is a singleton $\mathbb{P}$-a.s., i.e. synchronization holds.

Full support for the attractor

Definition

We say that φ is *swift transitive* if, for every closed ball $B(x,r)$ and every point y, there is a time $t>0$ such that

$$\mathbb{P}(\varphi_t(\cdot,B(x,r))\subset B(y,2r))>0.$$

Lemma

If φ is swift transitive and

$$\operatorname{ess\,inf}\{\operatorname{diam}(A(\omega));\omega\in\Omega\}=0 \qquad (*)$$

then

$$\mathbb{P}(A\subset U)>0$$

for every non-empty (deterministic) open set $U\subset E$.

Condition $(*)$ means that $\mathbb{P}(\operatorname{diam}(A)<\varepsilon)>0$ for every $\varepsilon>0$.

Full support for the attractor

Theorem

Assume that φ is asymptotically stable on some non-empty open set $U \subset X$ and is swift transitive. Let A satisfy

$$\operatorname{ess\,inf}\{\operatorname{diam}(A(\omega)); \omega \in \Omega\} = 0 \qquad (*)$$

Then A is a singleton, i.e. synchronization occurs.

Small diameter

Definition

We say that φ is *contracting on large sets* if for every $R > 0$, there is a ball $B(y, R)$ and a time $t > 0$ such that

$$\mathbb{P}\left(\mathrm{diam}\left(\varphi_t\left(\cdot, B\left(y, R\right)\right)\right) \leq \frac{R}{4}\right) > 0.$$

Lemma

Assume that φ is contracting on large sets and swift transitive. Then A has small diameter.

Examples

- How restrictive are the assumptions of asymptotic stability, swift transitivity and contraction on large sets?
- asymptotic stability:
    - Follows from local stable manifold theorem if $\lambda_{top} < 0$
    - For additive noise
      $$dX_t = b(X_t)dt + dW_t$$
      we have the bound
      $$\lambda_{top} \leq \int_{\mathbb{R}^d} \lambda^+(x) d\mu(x),$$
      with $\lambda^+(x) := \max_{|v|=1}(Db(x)v, v)$.
    - For gradient systems and small noise one often has $\lambda_{top} < 0$.
- swift transitivity: Satisfied basically for all SDE with additive noise.
- contraction on large sets: Assume b to be eventually monotone, i.e. there exists an $R > 0$ such that
  $$\langle b(x) - b(y), x - y \rangle \leq -c|x - y|^2$$
  for all $|x|, |y| \geq R$. Then contraction of large balls holds.

Gradient systems

Gradient systems

Gradient systems

- What can we say without assuming eventual monotonicity?
- Let φ be a white noise RDS and assume that P_t is ergodic with invariant measure μ.
- A random probability measure $\omega \mapsto \mu_\omega$ is a measurable function from Ω to the space of probability measures. We say that μ_ω is φ-invariant if

$$\varphi_t(\omega)_*\mu_\omega = \mu_{\theta_t \omega} \quad \text{a.s.}.$$

Fact

If μ_ω is an $\mathscr{F}_0$-measurable random invariant measure, then $\mu = \mathbb{E}\mu_\omega$ is P_t-invariant. Conversely, if μ is P_t-invariant then

$$\mu_\omega = \lim_{t\to\infty} \varphi_t(\theta_{-t}\omega)_*\mu$$

exists for $\mathbb{P}$-a.e. ω, it is an $\mathscr{F}_0$-measurable random invariant measure.

Gradient systems

Fact

Every random probability measure is supported by the weak random attractor, i.e.

$$\mu_\omega(A(\omega)) = 1 \quad a.s..$$

If φ is strongly mixing and $A(\omega) := supp(\mu_\omega)$ is compact then $A(\omega)$ is a (minimal) weak point attractor.

Lemma

The statistical equilibrium μ_ω is either discrete or diffuse. More precisely, either μ_ω consists of finitely many atoms of the same mass $\mathbb{P}$-a.s., i.e. there is an $N \in \mathbb{N}$ and $\mathscr{F}_0$-measurable random variables $a_1, \dots, a_N$ such that

$$\mu_\omega = \{\frac{1}{N}\delta_{a_i(\omega)} : i = 1, \dots, N\}$$

or μ_ω does not have point masses $\mathbb{P}$-a.s..

Gradient systems

Local stability can now be nicely captured in terms of the structure of the statistical equilibrium, i.e.

Lemma

Assume that φ is asymptotically stable on U with $\mu(U) > 0$. Then μ_ω is discrete.

Proposition

If φ is strongly mixing and asymptotically stable on U with $\mu(U) > 0$, then there is an $N \in \mathbb{N}$ and $\mathscr{F}_0$-measurable random variables $a_1, \ldots, a_N$ such that

$$A(\omega) = \operatorname{supp}(\mu_\omega) = \{a_i(\omega) : i = 1, \ldots, N\}$$

is a minimal weak point attractor.

Gradient systems

- It remains to show (under further assumptions) that trajectories get close. This replaces the assumption of eventual monotonicity/contraction of large balls.
- Let us consider gradient systems, i.e.

$$dX_t = -\nabla V(X_t)dt + \sigma dW_t$$

and assume strong mixing, i.e. $\rho(x) := e^{-\frac{2}{\sigma^2}V(x)} \in L^1(\mathbb{R}^d)$.

- To prove that trajectories get close, we need some kind of monotonicity of $b = -\nabla V$. From $\rho(x) \in L^1(\mathbb{R}^d)$ we get: For all $s \in S^{d-1}$, $\delta > 0$ there is a $z \in \mathbb{R}^d$ such that

$$\langle b(z) - b(z - \delta s), s \rangle < 0.$$

Gradient systems

Theorem

Assume that $\rho(x) := e^{-\frac{2}{\sigma^2}V(x)} \in L^1(\mathbb{R}^d)$ *and that* φ *is asymptotically stable on* U *with* $\mu(U) > 0$. *Then, there is a minimal weak point attractor* A *consisting of a single random point* $a(\omega)$ *and*

$$A(\omega) = supp(\mu_\omega) = \{a(\omega)\} \quad \mathbb{P}\text{-a.s.},$$

i.e. weak synchronization holds.

Question

Open questions:

- *For gradient systems: Does asymptotic stability always hold?*
- *What about the Lorenz system?*

Thanks

Thanks!