Synchronization by noise

Benjamin Gess

Department of Mathematics
University of Chicago

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Outline

1. Introduction
2. Background on random dynamical systems
3. Brief overview of known methods
   - Monotone RDS
   - Local stability + transitivity of the two-point motion
   - Perturbation techniques
4. A new approach to synchronization
5. Gradient systems
Introduction
Synchronization by noise

- We consider SDE on $\mathbb{R}^d$ of the type

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t. \quad (*)$$

- The inclusion of noise may simplify the long-time dynamics, i.e. while

$$dX_t = b(X_t)dt$$

may not be globally stable, the long-time behavior of (*) may be trivial.

- Roughly speaking: Synchronization by noise means that the random attractor consists of a single random point, i.e.

$$A(\omega) = \{a(\omega)\}, \quad \mathbb{P}\text{-a.s.}$$

- In particular: If synchronization occurs, then each two trajectories converge to each other in probability:

$$|X_t^x - X_t^y| \to 0 \quad \text{for } t \to \infty$$

in probability.
Model example

- Double-well potential, \( V(x) = -\frac{1}{2}|x|^2 + \frac{1}{4}|x|^4 \)

with additive Wiener noise, i.e.

\[
dX_t = (X_t - |X_t|^2 X_t) \, dt + \sigma \, dW_t.
\]
Model example

- Deterministic case ($\sigma = 0, \ d = 1$):

$$dX_t = (X_t - X_t^3)dt$$

- Attractor is given by closed unit ball: $A = \bar{B}_1(0) = [-1,1]$.
- Point attractor is given by $S^{d-1} \cup \{0\} = \{\pm 1, 0\}$.
- Simulation:
Model example

- Additive noise ($\sigma > 0$):
  \[ dX_t = (X_t - X_t^3)dt + \sigma dW_t \]

- Synchronization occurs: $A(\omega) = \{a(\omega)\}$ a.s.. In particular $|X_t^x - X_t^y| \to 0$ for $t \to \infty$ in probability.

- Simulation:
### Model example

- **Starting point of presented work:** How to prove this for $d > 1$?

#### Trajectories

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<tr>
<th>$\sigma = 1$</th>
<th>Distance of trajectories</th>
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Stochastic flows

- Consider
  \[ dX_t = b(X_t)dt + \sigma(X_t)dW_t. \] (*)
- By Arnold, Scheutzow, Kunita (among others) one may select a version \( \varphi_t \) of the solution \( X_t \) giving a random dynamical system

**Definition**

Let \( \theta := (\theta_t)_{t \in \mathbb{R}} \) be a **metric dynamical system** on \((\Omega, \mathcal{F}, \mathbb{P})\), i.e. \( \theta \) is a group of measurable and \( \mathbb{P} \)-preserving maps on \( \Omega \).

The map \( \varphi : [0, \infty) \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a **random dynamical system** (RDS) if

1. \( \varphi \) is measurable,
2. \( x \mapsto \varphi_t(\omega)x \) is continuous for all \( t \geq 0, \ \omega \in \Omega \),
3. \( \varphi_{t+s}(\omega)x = \varphi_t(\theta_s \omega)\varphi_s(\omega)x, \quad \varphi_0(\omega)x = x. \)

**Canonical setup:** \( \Omega = C(\mathbb{R}; \mathbb{R}^d) \), \( \mathbb{P} \) double-sided Wiener measure, \( \theta_t \omega(s) := \omega(t + s) - \omega(t) \), \( \varphi_t(\omega)x \) good version of a solution to (*) with i.c. \( x \).
White noise RDS

- For $s \leq t$ we define

$$\mathcal{F}_{s,t} := \sigma(\varphi_{s'}, t', s \leq s' \leq t' \leq t).$$

In particular, let $\mathcal{F}_0 := \mathcal{F}_{-\infty,0}$ be the past of the system up to time $0$.

**Definition**

An RDS $\varphi$ is a white noise RDS, if $\mathcal{F}_{s,t}, \mathcal{F}_{s',t'}$ are independent for all disjoint $(s, t), (s', t')$.

- Let $\varphi$ be a white noise RDS, then

$$P_t f(x) := \mathbb{E} f(\varphi_t(\cdot)x)$$

defines a Markov semigroup on $C_b(\mathbb{R}^d; \mathbb{R})$. 
Random attractors

Definition

A *weak random attractor* is a random set $A(\omega)$ such that

1. (invariance): $\varphi_t(\omega)A(\omega) = A(\theta_t \omega)$, a.s. for all $t \geq 0$.
2. (attraction):
   
   $$d(\varphi_t(\omega)B, A(\theta_t \omega)) \to 0 \quad \text{for } t \to \infty$$

   in probability, for each compact set $B$.
3. (compactness): $A(\omega)$ is compact a.s..

If we replace compact sets $B$ by points, then $A$ is called a *weak point attractor*.

Fact

*Weak random attractors are $\mathcal{F}_0$-measurable.*
Definition

We say that *synchronization* occurs if the weak random attractor is a singleton

\[ A(\omega) = \{a(\omega)\} \quad \text{a.s.} \]

We say that *weak synchronization* occurs if there is a singleton weak point attractor.
Brief overview of known methods

Brief overview of known methods
Known methods

There are several distinct methods to prove synchronization by noise available in the literature (there are many more!):

1. Monotone RDS + uniqueness of invariant measure (e.g. Arnold, Chueshov '98; Chuechov, Scheutzow '04)
2. Local stability + transitivity of the two-point motion (e.g. Baxendale '91)
3. Perturbation techniques/large deviation methods (e.g. Tearne '08, Martinelli, Scoppola, '88, '94)
4. Master-slave synchronization (Chueshov, Schmalfuss '10)
5. ... (many more) ...
Monotone RDS

- Monotone RDS: Assume that there is a partial order \( \leq \) of the state space (say \( \mathbb{R}^d \)) that is preserved by \( \varphi \), i.e.

  \[
  \text{if } x \leq y \text{ then } \varphi_t(\omega)x \leq \varphi_t(\omega)y.
  \]

- Assume that there is a random attractor \( A \) suitable "compatibility" of \( \leq \) with the topology on \( \mathbb{R}^d \). By Arnold, Chueshov '98 there are random variables \( a_-, a_+ \in A \) such that

  \[
  A(\omega) \subseteq [a_-(\omega), a_+(\omega)].
  \]

- Invariance of \( A \) implies that \( a_-, a_+ \) are invariant under \( \varphi \).

- Uniqueness of the invariant measure gives: \( \mathcal{L}(a_-) = \mathcal{L}(a_+) \) which implies \( a_- = a_+ \) a.s.

- Model example: For \( d = 1 \) this proves synchronization for

  \[
  dX_t = (X_t - X_t^3)dt + \sigma dW_t
  \]

  with \( \sigma > 0 \).
Local stability + transitivity of the two-point motion

- Assumptions: Local stability + transitivity of the two point motion
- Under suitable ergodic conditions, there is one number $\lambda_{top}$, called first (or top) Lyapunov exponent, such that
  \[ \lambda_{top} = \lim_{t \to \infty} \frac{1}{t} \log |D\varphi_t(x, \omega)v| \]
  exists for certain $x, v, \omega$ and it is the largest such limit.
- Local stability: $\lambda_{top} < 0$. This yields local stability, e.g. by local stable manifold theorem (e.g. Mohammed, Scheutzow '99)
- How to pass to global stability?
- Baxendale 91’: Assume transitivity of the two point motion
  \[ t \mapsto (\varphi_t(\omega)x, \varphi_t(\omega)y). \]
  In particular, i.e. for each $\delta > 0$ the stopping time
  \[ \tau := \inf\{t \geq 0 | |\varphi_t(\omega)x - \varphi_t(\omega)y| \leq \delta\} \]
  is finite with positive probability.
- For additive noise this is not a good assumption
  \[ d\varphi_t(x) = b(\varphi_t(x))dt + dW_t \]
  \[ d\varphi_t(y) = b(\varphi_t(y))dt + dW_t. \]
  The noise only shifts parallel to the diagonal.
Perturbation techniques

- Tearne '08 for small noise based on dynamical considerations.
- Consider:
  \[ dX_t = b(X_t)dt + \varepsilon dW_t \]
- Among other assumptions assume:
  - there are finitely many fixed points of \( b \)
  - all stable fixed points are hyperbolic
  - the noise is small enough.
Perturbation techniques

- The idea is:
  - each trajectory spends a long time in a basin of attraction, then jumps to another;
  - the trajectories of two different initial conditions may be in two different basins but sooner or later there is a fluctuation that sends both in the same basin and there they approach each other at least for small $\sigma$,
  - the time spent in such basins is so long that next fast transitions between basins cannot split again the particles.

- Only covers double well in $1 - d$:

The examples treated by Tearne ’08 are also covered by the general theory below.
Question

*Open question in the literature: Does synchronization occur for*

\[
dX_t = (X_t - |X_t|^2 X_t)dt + \sigma dW_t
\]

*with \( \sigma > 0 \) and \( d > 1 \)?*
A new approach to synchronization
Local stability

In the following let $\varphi$ be a white noise RDS, $(E, d)$ be a Polish space.

**Definition**

Let $U \subset E$ be a (deterministic) non-empty open set. We say that $\varphi$ is asymptotically stable on $U$ if there exists a (deterministic) sequence $t_n \uparrow \infty$ such that

$$\mathbb{P}\left( \lim_{n \to \infty} \text{diam}(\varphi_{t_n}(., U)) = 0 \right) > 0.$$ 

- We will see later that a negative top Lyapunov exponent implies asymptotic stability.

**Lemma**

*Let $\varphi$ be asymptotically stable on $U$ and assume* 

$$\mathbb{P}(A \subset U) > 0.$$ 

*Then $A$ is a singleton $\mathbb{P}$-a.s., i.e. synchronization holds.*
Full support for the attractor

Definition
We say that $\varphi$ is swift transitive if, for every closed ball $B(x,r)$ and every point $y$, there is a time $t > 0$ such that

$$\mathbb{P}(\varphi_t (\cdot, B(x,r)) \subset B(y,2r)) > 0.$$ 

Lemma
If $\varphi$ is swift transitive and

$$\text{ess inf} \{\text{diam}(A(\omega)); \omega \in \Omega\} = 0 \quad (*)$$

then

$$\mathbb{P}(A \subset U) > 0$$

for every non-empty (deterministic) open set $U \subset E$.

Condition (*) means that $\mathbb{P}(\text{diam}(A) < \epsilon) > 0$ for every $\epsilon > 0$. 
Full support for the attractor

**Theorem**

Assume that $\varphi$ is asymptotically stable on some non-empty open set $U \subset X$ and is swift transitive. Let $A$ satisfy

$$\text{ess inf} \{\text{diam}(A(\omega)); \omega \in \Omega\} = 0 \quad (*)$$

Then $A$ is a singleton, i.e. synchronization occurs.
Small diameter

Definition
We say that $\phi$ is \textit{contracting on large sets} if for every $R > 0$, there is a ball $B(y, R)$ and a time $t > 0$ such that

$$\mathbb{P}\left(\text{diam} (\phi_t (\cdot, B(y, R))) \leq \frac{R}{4}\right) > 0.$$  

Lemma
\textit{Assume that $\phi$ is contracting on large sets and swift transitive. Then $A$ has small diameter.}
Examples

- How restrictive are the assumptions of asymptotic stability, swift transitivity and contraction on large sets?
  - asymptotic stability:
    - Follows from local stable manifold theorem if $\lambda_{top} < 0$
    - For additive noise
      \[ dX_t = b(X_t)dt + dW_t \]
      we have the bound
      \[ \lambda_{top} \leq \int_{\mathbb{R}^d} \lambda^+(x)d\mu(x), \]
      with $\lambda^+(x) := \max_{|v|=1} (Db(x)v, v)$.
    - For gradient systems and small noise one often has $\lambda_{top} < 0$.
  - swift transitivity: Satisfied basically for all SDE with additive noise.
  - contraction on large sets: Assume $b$ to be eventually monotone, i.e. there exists an $R > 0$ such that
    \[ \langle b(x) - b(y), x - y \rangle \leq -c|x - y|^2 \]
    for all $|x|, |y| \geq R$. Then contraction of large balls holds.
Gradient systems
Gradient systems

- What can we say without assuming eventual monotonicity?
- Let $\varphi$ be a white noise RDS and assume that $P_t$ is ergodic with invariant measure $\mu$.
- A random probability measure $\omega \mapsto \mu_\omega$ is a measurable function from $\Omega$ to the space of probability measures. We say that $\mu_\omega$ is $\varphi$-invariant if
  \[ \varphi_t(\omega) \ast \mu_\omega = \mu_{\theta_t \omega} \quad \text{a.s..} \]

**Fact**

If $\mu_\omega$ is an $\mathcal{F}_0$-measurable random invariant measure, then $\mu = \mathbb{E}\mu_\omega$ is $P_t$-invariant. Conversely, if $\mu$ is $P_t$-invariant then
\[ \mu_\omega = \lim_{t \to \infty} \varphi_t(\theta_{-t} \omega) \ast \mu \]
exists for $\mathbb{P}$-a.e. $\omega$, it is an $\mathcal{F}_0$-measurable random invariant measure.
Fact

Every random probability measure is supported by the weak random attractor, i.e.

\[ \mu_\omega(A(\omega)) = 1 \quad \text{a.s.} \]

If \( \phi \) is strongly mixing and \( A(\omega) := \text{supp}(\mu_\omega) \) is compact then \( A(\omega) \) is a (minimal) weak point attractor.

Lemma

The statistical equilibrium \( \mu_\omega \) is either discrete or diffuse. More precisely, either \( \mu_\omega \) consists of finitely many atoms of the same mass \( \mathbb{P}\)-a.s., i.e. there is an \( N \in \mathbb{N} \) and \( \mathcal{F}_0 \)-measurable random variables \( a_1, \ldots, a_N \) such that

\[ \mu_\omega = \left\{ \frac{1}{N} \delta_{a_i}(\omega) : i = 1, \ldots, N \right\} \]

or \( \mu_\omega \) does not have point masses \( \mathbb{P}\)-a.s..
Gradient systems

Local stability can now be nicely captured in terms of the structure of the statistical equilibrium, i.e.

**Lemma**

*Assume that \( \varphi \) is asymptotically stable on \( U \) with \( \mu(U) > 0 \). Then \( \mu_\omega \) is discrete.*

**Proposition**

*If \( \varphi \) is strongly mixing and asymptotically stable on \( U \) with \( \mu(U) > 0 \), then there is an \( N \in \mathbb{N} \) and \( \mathcal{F}_0 \)-measurable random variables \( a_1, \ldots, a_N \) such that

\[
A(\omega) = \text{supp}(\mu_\omega) = \{a_i(\omega) : i = 1, \ldots, N\}
\]

is a minimal weak point attractor.*
Gradient systems

- It remains to show (under further assumptions) that trajectories get close. This replaces the assumption of eventual monotonicity/contraction of large balls.
- Let us consider gradient systems, i.e.

\[ dX_t = -\nabla V(X_t)dt + \sigma dW_t \]

and assume strong mixing, i.e. \( \rho(x) := e^{-\frac{2}{\sigma^2}V(x)} \in L^1(\mathbb{R}^d) \).
- To prove that trajectories get close, we need some kind of monotonicity of \( b = -\nabla V \). From \( \rho(x) \in L^1(\mathbb{R}^d) \) we get: For all \( s \in S^{d-1}, \delta > 0 \) there is a \( z \in \mathbb{R}^d \) such that

\[ \langle b(z) - b(z - \delta s), s \rangle < 0. \]
Gradient systems

**Theorem**

Assume that \( \rho(x) := e^{-\frac{2}{\sigma^2} V(x)} \in L^1(\mathbb{R}^d) \) and that \( \varphi \) is asymptotically stable on \( U \) with \( \mu(U) > 0 \). Then, there is a minimal weak point attractor \( A \) consisting of a single random point \( a(\omega) \) and

\[
A(\omega) = \text{supp}(\mu_\omega) = \{a(\omega)\} \quad \mathbb{P}\text{-a.s.,}
\]

i.e. weak synchronization holds.

**Question**

Open questions:

- For gradient systems: Does asymptotic stability always hold?
- What about the Lorenz system?
Thanks

Thanks!