

Finite speed of propagation for stochastic porous media equations.

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Outline

- 1 The deterministic case
- 2 The stochastic case
- 3 Infinite dimensional random attractors
 - Deterministic case
 - Stochastic case

The deterministic case

The deterministic case

The Deterministic PME

- Recall: The deterministic porous medium equation

$$\frac{d}{dt}u = \Delta(u^m), \quad m > 1 \quad (1)$$

for non-negative initial conditions $u_0 \geq 0$ [Vázquez, 2006]. Note that for simplicity we will always write u^m for $|u|^{m-1}u$, even if u not necessarily non-negative.

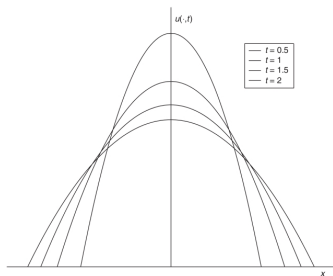
- In the superlinear case $m > 1$, (1) has degenerate diffusivity:

$$\frac{d}{dt}u = mu^{m-1}\Delta u + m(m-1)u^{m-2}|\nabla u|^2.$$

The diffusivity coefficient vanishes for $u \rightarrow 0$.

Finite speed of propagation

- Limited regularity: ∇u discontinuous. E.g. Barenblatt solutions:



- Regularity is limited precisely at the free boundary.
- Comparison to Barenblatt solutions yields finite speed of propagation.

Optimal estimates

- Finite speed of propagation is a local property.

- *Finite speed of hole-filling:*

Let u_0 vanish in $B_R(x_0)$. Then u vanishes in $B_{R(t)}(x_0)$ with

$$R(t) \geq R - \left(\frac{\|u\|_\infty^{m-1}}{C_{det}} t \right)^{1/2}.$$

This is optimal.

- *Finite speed of propagation:*

Let $S(t) := \{x \in \mathbb{R}^d \mid u(t, x) > 0\}$ be the positivity set. Then

$$S(t+h) \subseteq B_{ch^{1/2}}(S(t)).$$

The stochastic case

The stochastic case

Stochastic porous medium equation

- Stochastic porous medium equation

$$dX_t = \Delta X_t^m dt + \sum_{k=1}^N f_k X_t \circ d\beta_t^{(k)}. \quad (\text{SPME})$$

on bounded domains, homogeneous Dirichlet boundary conditions.

- β^k Brownian motion, $f_k \in C^\infty(\mathcal{O})$.

Known results

- *Finite speed of hole-filling* [Barbu, Röckner, EJP 2012]:
Let X_0 vanish in $B_R(x_0)$. Then X vanishes in $B_{R(t,\omega)}(x_0)$ for some function $R(\cdot, \omega) : [0, T] \rightarrow (0, R)$.
- No uniform control on $R(t, \omega)$ in $x_0 \rightarrow$ cannot deduce finite speed of propagation
- No information about optimality of the bounds
- Two aims:
 - show finite speed of propagation
 - deduce (locally) optimal bounds

Transformation

- Recall:

$$dX_t = \Delta(X_t^m) dt + \sum_{k=1}^N f_k X_t \circ d\beta_t^{(k)}.$$

- Set $Y_t := e^{\mu t} X_t$, where $\mu_t = -\sum_{k=1}^N f_k \beta_t^{(k)}$. Then

$$\partial_t Y(t, x) = e^{\mu(t, x)} \Delta \left(e^{-\mu(t, x)} Y(t, x) \right)^m.$$

- Existence and uniqueness in [Barbu, Röckner, JDE, 2011], [G., arXiv:1108.2413; to appear in AoP].

Non-spatially distributed noise

- Assume f_k constant. Then

$$\partial_t Y_t = e^{(1-m)\mu t} \Delta Y_t^m.$$

- Let $F' := e^{(1-m)\mu t}$, $g = F^{-1}$. Then $u_t := Y_{g(t)}$ solves

$$\frac{d}{dt} u = \Delta(u^m)$$

- Finite speed of propagation follows from the deterministic case and the estimates are **optimal**.
- Finite speed of hole-filling:*

Let X_0 vanish in $B_R(x_0)$. Then X_t vanishes in $B_{R_{stoch}(t,\omega)}(x_0)$ with

$$\begin{aligned} R_{stoch}(t) &= R - \left(\frac{H^{m-1}}{C_{det}} F(t) \right)^{\frac{1}{2}} \\ &= R - \left(\frac{H^{m-1}}{C_{det}} \int_0^t e^{-(m-1)\mu_r(\omega)} dr \right)^{\frac{1}{2}}. \end{aligned}$$

Spatially distributed noise

- Recall:

$$\partial_t Y(t, x) = e^{\mu(t, x)} \Delta \left(e^{-\mu(t, x)} Y(t, x) \right)^m.$$

- freeze coefficients in space:

$$\partial_t Y(t, x) \approx e^{\mu(t, x_0)} \Delta \left(e^{-\mu(t, x_0)} Y(t, x) \right)^m,$$

on small balls $B_r(x_0)$.

- freeze coefficients in time:

$$\partial_t Y(t, x) \approx \Delta Y(t, x)^m,$$

for small times $t \approx 0$.

Hole-filling

- *Finite speed of hole-filling:*

Let X_0 vanish in $B_R(x_0)$. Then X_t vanishes in $B_{R_{stoch}(t,\omega)}(x_0)$ with

$$\begin{aligned} R_{stoch}(t, \omega) &= R - \left(\frac{H^{m-1}}{C_{det}} F(t, \omega) \right)^{\frac{1}{2}} C_R^{-\frac{1}{2}}(\omega) \\ &= R - \left(\frac{H^{m-1}}{C_{det}} \int_0^t e^{-(m-1)\mu_r(x_0, \omega)} dr \right)^{\frac{1}{2}} C_R^{-\frac{1}{2}}(\omega), \end{aligned}$$

with $\lim_{R \downarrow 0} C_R = 1$.

- For $R \approx 0$ we recover the optimal rate from the spatially homogeneous case with $\mu_r \equiv \mu_r(\xi_0)$.

Hole-filling

- *Finite speed of hole-filling:*

Let X_0 vanish in $B_R(x_0)$. Then X_t vanishes in $B_{R_{stoch}(t,\omega)}(x_0)$ with

$$R_{stoch}(t, \omega) = R - \left(\frac{H^{m-1}}{C_{det}} t \right)^{\frac{1}{2}} \sqrt{C_t(\omega)}.$$

with $\lim_{t \downarrow 0} C_t = 1$.

- For $t \approx 0$ we recover the optimal rate from the deterministic case.

Finite speed of propagation

- *Finite speed of propagation:*

Let X be an essentially bounded, non-negative solution to (SPME). Then,

$$\text{supp}(X_t) \subseteq B_{\sqrt{t} \left(\frac{H^{m-1}}{C_{\det}} \right)^{\frac{1}{2}} \sqrt{C_t(\omega)}}(\text{supp}(X_0)), \quad \forall t \in [0, T],$$

with $C_t \rightarrow 1$ for $t \rightarrow 0$.

Infinite dimensional random attractors

Infinite dimensional random attractors

Deterministic case

- Linear perturbation

$$\frac{d}{dt}u = \Delta(u^m) + \lambda u, \quad (2)$$

with $\lambda \geq 0$, bounded domain, homogeneous Dirichlet boundary conditions.

- The attractor to (2) has infinite fractal dimension iff $\lambda > 0$ [Efendiev, Zelik, Proc. Lond. Math. Soc., 2008].

Additive noise

- PME perturbed by additive noise

$$dX_t = (\Delta(X_t^m) + \lambda X_t)dt + \sigma dW_t. \quad (3)$$

- If $d \leq 2$, noise “non-degenerate”, then (3) has a random attractor consisting of a single (random) point [G., JDDE, 2013].

Multiplicative noise

- PME perturbed by linear multiplicative noise

$$dX_t = (\Delta(X_t^m) + \lambda X_t) dt + \sum_{k=1}^N f_k X_t \circ d\beta_t^{(k)}. \quad (4)$$

- The random attractor for (4) has infinite fractal dimension iff $\lambda > 0$.
- Cannot proceed via linearization
- Unstable manifold at 0:

$$\mathcal{M}^+(0, \omega) := \{u_0 \in X \mid \exists u : (-\infty, 0] \rightarrow X, \text{ such that } \varphi(t; \theta_{-t}\omega)u(-t) = u_0 \\ \text{for all } t \geq 0 \text{ and } \|u(t)\|_{L^\infty(\mathcal{O})} \rightarrow 0 \text{ for } t \rightarrow -\infty\}.$$