Stochastic dynamics induced by porous media equations with space-time linear multiplicative noise

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with $m > 1$, Dirichlet boundary conditions, driving signals $z^{(k)} \in C([0,T];\mathbb{R})$, $f_k \in C^\infty(\bar{\mathcal{O}})$.

Overview

1. Construction of a stochastic flow
   (a) Pathwise solution
   (b) Random case
2. Existence of Random Attractors
   (a) Bounded absorption
   (b) Asymptotic compactness & regularity of solutions

Part 1: Generation of Stochastic Flows

Recall

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With $\mu_t(\xi) = -\sum_{k=1}^N f_k(\xi) z_t^{(k)}$ the transformation $Y = e^{\mu} X$ yields:

$$dY_t = e^{\mu_t}\Delta(e^{-m\mu_t}|Y_t|^m sgn(Y_t)). \qquad \text{(TPME)}$$

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Partial construction: [BR11][1].

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Let $Y_0 \in L^1(\mathcal{O})$, $z \in C([0,T];\mathbb{R}^N)$. *There is a unique* $Y \in C([0,T];L^1(\mathcal{O}))$ *satisfying*

- $Y \in L^\infty([\tau,T]\times\mathcal{O})$, $\forall \tau > 0$ *and*
$$\frac{dY}{dt} = e^{\mu_t}\Delta(e^{-m\mu_t}Y_t^m)$$
for a.e. $t \in [0,T]$ *as an equation in* $H := (H_0^1(\mathcal{O}))^*$.

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for a.e. $t \in [0,T]$ *as an equation in* $H := (H_0^1(\mathcal{O}))^*$.
- L^1*-contractivity:*
$$\sup_{t\in[0,T]} \|Y_t^{(1)} - Y_t^{(2)}\|_{L^1(\mathcal{O})} \le C\|Y_0^{(1)} - Y_0^{(2)}\|_{L^1(\mathcal{O})}.$$

Random Case: Generation of RDS

Let $(z_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^N$-valued stochastic process and $((\Omega,\mathcal{F},\mathbb{P}),(\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system.

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$(S1)$ $z_t(\omega)-z_s(\omega)=z_{t-s}(\theta_s\omega)-z_0(\theta_s\omega)$, $\forall t,s\in\mathbb{R}$, $\omega\in\Omega$.

$(S2)$ z_t has continuous paths.

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- φ *is quasi-weakly-continuous on* $L^p(\mathcal{O})$, $p \in [1,\infty)$ *and quasi-weakly***-continuous on* $L^\infty(\mathcal{O})$.
- φ *is order preserving (i.e.* $\varphi(t,\omega)x_1 \le \varphi(t,\omega)x_2$, *a.e. in* $\mathcal{O}$ *if* $x_1, x_2 \in L^1(\mathcal{O})$ *with* $x_1 \le x_2$ *a.e. in* $\mathcal{O}$*).*

Part 2: Random Attractors

Bounded absorption

Solution: Construct explicit supersolution with initial value ∞ and bounded for all $t > 0$. The construction combines an interval splitting technique from [BR11][2] and the known deterministic case:

$$K(t,\xi) = At^{-\frac{1}{m-1}}(R^2 - |\xi|^2)^{\frac{1}{m}}.$$

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Theorem

There is a function $U : (0,T] \to \mathbb{R}$ ($U(0) \equiv \infty$), piecewise smooth on $(0,T]$ such that

$$|Y_t| \le U_t, \text{ a.e. in } \mathcal{O},$$

for all $t \in [0,T]$, independent of the initial condition $Y_0 \in L^1(\mathcal{O})$.

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Asymptotic Compactness

We say that a quantity depends only on the data if it is a function of d, m, T, $\|X_0\|_{L^\infty(\mathcal{O})}$.

Theorem

Let $z \in C([0,T];\mathbb{R}^N)$, $X_0 \in L^1(\mathcal{O})$ and X be the corresponding solution. Then

- *X is continuous on every compact set $K \subseteq (0,T] \times \mathcal{O}$, with modulus of continuity depending only on the data and $dist(K, \partial\mathcal{O}_T)$.*
- *If $\mathcal{O}$ is uniformly convex, then X is continuous on $[\tau, T] \times \bar{\mathcal{O}}$ with modulus of continuity depending only on the data, θ^*, τ.*

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The RDS φ has a random attractor A as an RDS on $L^1(\mathcal{O})$. A is compact and attracting in each $L^p(\mathcal{O})$, $p \in [1, \infty)$.

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