Random attractors for stochastic porous media equations perturbed by space-time linear multiplicative noise

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Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded domain, we consider

$$dX_t = \Delta(|X_t|^m sgn(X_t))dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \qquad \text{(RPME)}$$
$$X(0) = X_0,$$

$m > 1$, with Dirichlet boundary conditions, driving signals $z^{(k)} \in C([0,T];\mathbb{R})$, $f_k \in C^\infty(\bar{\mathcal{O}})$.

Overview

1. Construction of a stochastic flow
   (a) Pathwise solution
   (b) Random case
2. Existence of Random Attractors
   (a) Bounded absorption
   (b) Asymptotic compactness & regularity of solutions

Part 1: Generation of Stochastic Flows

Recall

$$dX_t = \Delta(|X_t|^m sgn(X_t))dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \text{ on } \mathcal{O}_T.$$

With $\mu_t(\xi) = -\sum_{k=1}^{N} f_k(\xi) z_t^{(k)}$ the transformation $Y = e^{\mu} X$ yields:

$$dY_t = e^{\mu_t} \Delta(e^{-m\mu_t} |Y_t|^m sgn(Y_t)). \qquad \text{(TPME)}$$

Partial construction: [BR10][1].

[1] V. Barbu, M. Röckner, *On a random scaled Porous Medium Equation, preprint, 2010*

Pathwise Solution

Theorem

Essentially bounded distributional solutions to (TPME) *are unique.*

Theorem

Let $Y_0 \in L^1(\mathcal{O})$, $z \in C([0,T];\mathbb{R}^N)$. *There is a unique map* $Y \in C([0,T];L^1(\mathcal{O}))$ *satisfying*

1. $Y \in L^\infty([\tau,T]\times\mathcal{O})$, $\forall \tau > 0$ *and*

$$\frac{dY}{dt} = e^{\mu_t}\Delta\Phi(e^{-\mu_t}Y_t)$$

   for a.e. $t \in [0,T]$ *as an equation in* $H := (H_0^1(\mathcal{O}))^*$.

2. L^1*-contractivity:*

$$\sup_{t\in[0,T]} \|(Y_t^{(1)} - Y_t^{(2)})^+\|_{L^1(\mathcal{O})} \le C\|(Y_0^{(1)} - Y_0^{(2)})^+\|_{L^1(\mathcal{O})}.$$

Random Case: Generation of RDS

Let $(z_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^N$-valued stochastic process and $((\Omega,\mathcal{F},\mathbb{P}),(\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system. Assume

(S1) $z_t(\omega)-z_s(\omega)=z_{t-s}(\theta_s\omega)-z_0(\theta_s\omega)$, $\forall t,s\in\mathbb{R}$, $\omega\in\Omega$.

(S2) z_t has continuous paths.

Theorem

1. $\varphi(t,\omega)x := X(t,0;\omega)x = e^{-\mu_t} Y(t,0;\omega)x$ *defines a continuous RDS on* $L^1(\mathcal{O})$.
2. φ *is quasi-weakly-continuous on* $L^p(\mathcal{O})$, $p \in [1,\infty)$ *and quasi-weakly*-continuous on* $L^\infty(\mathcal{O})$.
3. φ *is order preserving (i.e.* $\varphi(t,\omega)x_1 \leq \varphi(t,\omega)x_2$, *a.e. in* $\mathcal{O}$ *if* $x_1, x_2 \in L^1(\mathcal{O})$ *with* $x_1 \leq x_2$ *a.e. in* $\mathcal{O}$*).*

Part 2: Random Attractors

Bounded absorption

Solution: Construct explicit supersolution with initial value ∞ and bounded for all $t > 0$. The construction combines an interval splitting technique from [BR10][2] and the known deterministic case:

$$K(t,\xi) = At^{-\frac{1}{m-1}}(R^2 - |\xi|^2)^{\frac{1}{m}}.$$

Theorem

There is a function $U : [0,T] \to \bar{\mathbb{R}}$ *(*$U(0) \equiv \infty$*), piecewisely smooth on* $(0,T]$ *such that*

$$|Y_t| \le U_t, \text{ a.e. in } \mathcal{O},$$

for all $t \in [0,T]$*, independent of the initial condition* $Y_0 \in L^1(\mathcal{O})$*.*

[2] V. Barbu, M. Röckner, *On a random scaled Porous Medium Equation, preprint, 2010*

Asymptotic Compactness

We say that a quantity depends only on the data if it is a function of d, m, T, $\|X_0\|_{L^\infty(\mathcal{O})}$.

Theorem

Let $z \in C([0,T];\mathbb{R}^N)$, $X_0 \in L^1(\mathcal{O})$ and X be the corresponding solution. Then

1. *X is continuous on every compact set $K \subseteq (0,T] \times \mathcal{O}$, with modulus of continuity depending only on the data and dist$(K, \partial\mathcal{O}_T)$.*
2. *If $\mathcal{O}$ is uniformly convex, then X is continuous on $[\tau, T] \times \bar{\mathcal{O}}$ with modulus of continuity depending only on the data, θ^*, τ.*

Theorem

The RDS φ has a random attractor A as an RDS on $L^1(\mathcal{O})$. A is compact and attracting in each $L^p(\mathcal{O})$, $p \in [1, \infty)$.

$A(\omega)$ is a bounded set in $L^\infty(\mathcal{O})$ and the functions in $A(\omega)$ are equicontinuous on every compact set $K \subseteq \mathcal{O}$.

If $\mathcal{O}$ is uniformly convex, then $A(\omega)$ is compact and attracting in $L^\infty(\mathcal{O})$.