Random Attractors for Stochastic Porous Media Equations

Benjamin Gess
University of Bielefeld

joint work with Wolf-Jürgen Beyn
Paul Lescot
Michael Röckner

published in Communications in Partial Differential Equations, 2011
(Volume 36,Issue 3).

Outline

Basics on Random Dynamical Systems (RDS)

Definition (1.1)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta_t : \Omega \to \Omega$, $t \in \mathbb{R}$, be a family of maps. Then $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ is said to be a **metric dynamical system** if

- $\theta : \mathbb{R} \times \Omega \to \Omega$ is measurable
- $\theta_0 = id$, $\theta_{t+s} = \theta_t \circ \theta_s$
- $(\theta_t)_* \mathbb{P} = \mathbb{P}$

e.g. $\Omega = C_0(\mathbb{R}; \mathbb{R})$, $\mathbb{P}$ = Wiener measure, $\theta_t(\omega) = \omega(t + \cdot) - \omega(t)$ ("Wiener shift").

Basics on Random Dynamical Systems (RDS)

Definition (1.2)

Let (H, d) be a complete and separable metric space, $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ a metric dynamical system and $\varphi : \mathbb{R}_+ \times \Omega \times H \to H$ measurable with

- $\varphi(0, \omega) = \text{id}$
- $\varphi(t+s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega)$ (cocycle property)
- $\varphi(t, \omega) : H \to H$ continuous.

Then φ is called a **random dynamical system** (RDS).

Definition (1.3)

- $K : \Omega \to 2^H$ is called **measurable** if $\omega \mapsto d(x, K(\omega))$ is measurable for all $x \in H$, where d is the Hausdorff semidistance. K is also called a **random set**.

Basics on Random Dynamical Systems (RDS)

Definition (1.3)

- Let A, B be random sets. A is said to **absorb** B if for $\mathbb{P}$-a.e. $\omega \in \Omega$ there exists an **absorption time** $t_B(\omega)$ such that for all $t \geq t_B(\omega)$

$$\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subseteq A(\omega).$$

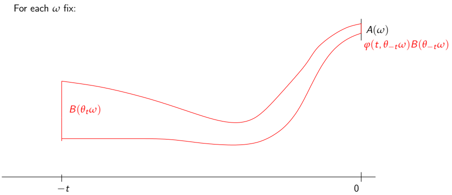

Figure: pullback absorption

Basics on Random Dynamical Systems (RDS)

Definition (1.3)

- A is said to **attract** B if

$$d(\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)) \xrightarrow[t\to\infty]{} 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

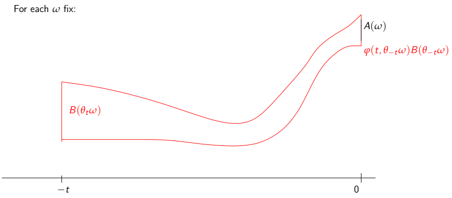

Figure: pullback attraction

Basics on Random Dynamical Systems (RDS)

Definition (1.4)

A **random attrator** for an RDS φ is a random set A satisfying $\mathbb{P}$-a.s.

- (compactness): $A(\omega)$ is compact.
- (invariance): $\varphi(t,\omega)A(\omega) = A(\theta_t\omega)$ for all $t > 0$.
- (attraction): A attracts all deterministic bounded sets $B \subseteq H$, i.e.

$$d(\varphi(t,\theta_{-t}\omega)B, A(\omega)) \xrightarrow[t\to\infty]{} 0.$$

RDS given by Stochastic Porous Media Equations

Consider a stochastic porous media equation over a bounded open set $\Lambda \subset \mathbb{R}^d$

$$dX_t = \Delta(\Phi(X_t))\, dt + Q dW_t, \ t \geq s, \qquad \text{(SPME)}$$

where $t, s \in \mathbb{R}$ and for $m \in \mathbb{N}$

$$QW_t = \sum_{j=1}^{m} \varphi_j \beta_j,$$

$\varphi_1, \ldots, \varphi_m \in C_0^1(\Lambda)$, $\beta_1, \ldots, \beta_m$ independent $\mathbb{R}$-valued Brownian motions on canonical Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ (i.e. $\Omega = C(\mathbb{R}; \mathbb{R}^m)$), and

RDS given by Stochastic Porous Media Equations

- $\Phi : \mathbb{R} \to \mathbb{R}$ continuous, $\Phi(0) = 0$ (for simplicity)
- Φ monotone
- $\exists p \geq 1$, $a > 0$, $c \geq 0$ such that

$$\Phi(s)s \geq a|s|^{p+1} - c \quad \forall s \in \mathbb{R} \text{ (“coercive”);}$$

- $\Phi(s) \leq \text{const.}(|s|^p + 1) \quad \forall s \in \mathbb{R}$ (“polynomial growth”).

RDS given by Stochastic Porous Media Equations

Consider the triple

$$V := L^{p+1}(\Lambda) \subset H \subset V^*,$$

where $H := (H_0^1(\Lambda))^*$. By "shifting the noise" (SPME) reduces to deterministic porous media equation (PME) with time and ω-dependent coefficients, i.e. change of variables

$$X_t \rightsquigarrow X_t - QW_t =: Y_t$$

yields (in integral form) for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$Y_t(\omega) = x - QW_s(\omega) + \int_s^t A_\omega(r, Y_r(\omega))dr, \quad t \geq s, \quad \text{(PME)}$$

where $x \in H$ and

$$A_\omega(r, \cdot) := \Delta\Phi(\cdot + QW_r(\omega)).$$

RDS given by Stochastic Porous Media Equations

By standard variational methods (PME) has a unique solution for each fix $\omega \in \Omega$:

$$Y(\cdot, s, x, \omega) \in L^{p+1}_{\text{loc}}([s, \infty); V) \cap C([s, \infty), H).$$

Then reversing "shift-transformation"

$$X_t(\omega) := Y(t, s, x, \omega) + QW_t(\omega) \quad , \quad t \geq s; \; t, s \in \mathbb{R},$$

is stochastic process which solves

$$X_t = x + \int_s^t \Delta\Phi(X_r)dr + QW_t - QW_s. \qquad \text{(SPME)}$$

Then

$$\varphi(t, \omega)x := X(t, 0, \omega)x, \quad t \geq 0,$$

defines an RDS.

RDS given by Stochastic Porous Media Equations

Hypothesis (Assumptions 1)

Assume $\varphi_j \in H_0^{2,p+1}(\Lambda)$*, thus* $QW_t \in H_0^{2,p+1}(\Lambda)$*. Let further* $\zeta : \mathbb{R} \to \mathbb{R}$*,* $\zeta(0) = 0$ *be a function such that*

- ζ*-Weak monotonicity: For all* $t, s \in \mathbb{R}$

$$(\Phi(t) - \Phi(s))(t - s) \geq (\zeta(t) - \zeta(s))^2.$$

- ζ*-Coercivity: For* p, a, c *as in (A2) and for all* $s \in \mathbb{R}$

$$\Phi(s)s \geq \zeta(s)^2 \geq a|s|^{p+1} - c.$$

RDS given by Stochastic Porous Media Equations

Hypothesis (Assumptions 2)

Let $\varphi_j \in H_0^{1,p+1}(\Lambda)$. *Assume further that* $\Phi \in C^1(\mathbb{R})$, *satisfies*

- *Coercivity:* $\exists p \geq 1, a > 0, c \geq 0$ *such that*

$$\left(\int_0^s \sqrt{\Phi'(r)}dr\right)^2 \geq a|s|^{p+1} - c \quad \forall s \in \mathbb{R},$$

- *Strict monotonicity:* $\Phi'(r) > 0$ *for almost all* $r \in \mathbb{R}$,
- *Polynomial growth:* $\Phi'(s) \leq const(|s|^{p-1} + 1) \quad \forall s \in \mathbb{R}$.

Remark With $\zeta(s) := \int_0^s \sqrt{\Phi'(r)}dr, s \in \mathbb{R}$, assumptions 2 imply assumptions 1.

Example: $\Phi(s) = s|s|^{p-1}$ for $p \geq 1$. Then $\zeta(s) = \frac{2\sqrt{p}}{p+1} s|s|^{\frac{p-1}{2}}$.

Main Results

Theorem (3.1)

Under the above conditions on (QW and) Φ the RDS given by (SPME) as defined above has a random attractor.

Main Results

Theorem (3.2)

Assume that (QW_t) is a V-valued Wiener process and that $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous satisfying:

- $\exists p > 1, \eta > 0$ *such that*

$$(t-s)(\Phi(t) - \Phi(s)) \geq \eta |t-s|^{p+1} \quad \forall s, t \in \mathbb{R}.$$
$$|\Phi(s)| \leq const.(1 + |s|^p) \quad \forall s \in \mathbb{R}.$$

(Example: $\Phi(s) = s|s|^{p-1}$.)
Then the random attractor given by (SPME) as defined above is given by

$$A(\omega) := \lim_{s \to -\infty} X(0, s, \omega)x,$$

(which is independent of x!), i.e. is a random point.

Idea of Proof

Definition (1.5)

For a random set A we define the Ω-**limit** to be

$$\Omega_A(\omega) := \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} \varphi(t, \theta_{-t}\omega) A(\theta_{-t}\omega)}, \ \omega \in \Omega.$$

Proposition (1.6 [Crauel/Flandoli: PTRF 1994])

Let φ be an RDS and assume the existence of a compact random set K absorbing every deterministic bounded set $B \subseteq H$. Then there exists a random attractor A and it is given by

$$A(\omega) = \overline{\bigcup_{B \subseteq H, \ B \text{ bounded}} \Omega_B(\omega)}.$$

Idea of Proof

Steps

1. Bounded absorption: Bound for the H-norm
2. Compact absorption: Bound for the stonger L^2-norm

Idea of Proof

Crucial estimates:
$\exists \alpha > 0$ and polynomials p_1 and p_2 with random coefficients such that

$$\frac{d}{dt}||Y_t||_H^2 \leq -\alpha||X_t||_{L^2}^2 + p_1(t). \qquad \circledast$$

$$\frac{d}{dt}||Y_t||_{L^2}^2 \leq -\alpha||Y_t||_{L^2}^2 + p_2(t), \qquad \circledast\circledast$$

Remark $\circledast\circledast$ in particular implies that $\forall \omega \in \Omega_0, \mathbb{P}(\Omega_0) = 1$,
$t \to Y_t(\omega)$ right continuous in $L^2(\Lambda)$.
Higher regularity:

$$\Phi(X(\cdot, s, \omega)x) \in L_{loc}^{\frac{p+1}{p}}([s, \infty); H_0^{1, \frac{p+1}{p}}),$$

for $x \in L^2(\Lambda)$, $s \in \mathbb{R}$ and $\omega \in \Omega$