Random Attractors for Monotone SPDEs

Benjamin Gess

International Graduate College (IGK),
"Stochastics and Real World Models",
Department of Mathematics,
Bielefeld University

IGK Seminar
17th January 2010
Bielefeld

joint work with: Wei Liu, Michael Röckner

published in: Journal of Differential Equations, March 2011.

1. Introduction
   - Motivation
   - Basics on RDS
   - Known results
   - Variational framework for SPDE

2. Main results
   - Generation of RDS
   - Single point attraction
   - Existence of the random attractor

3. Applications

'Fully developed' dynamics (i.e. that have been run for a long time) have reduced complexity. There exists a 'small' set $A \subseteq H$ (H the state space) such that

- A is invariant under the flow
- Each trajectory can be approximated arbitrarily well by one lying in A (for large times)
- A is minimal.

Sets with these properties are called attractors.

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta_t : \Omega \to \Omega, t \in \mathbb{R}$ be a family of maps. Then $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ is said to be a metric dynamical system if

- $\theta : \mathbb{R} \times \Omega \to \Omega$ is measurable
- $\theta_0 = id$, $\theta_{t+s} = \theta_t \circ \theta_s$
- $(\theta_t)_* \mathbb{P} = \mathbb{P}$

e.g. $\Omega = C_0(\mathbb{R}; \mathbb{R})$, $\mathbb{P} = \gamma$, $\theta_t(\omega) = \omega(t + \cdot) - \omega(t)$.

Definition

Let (H, d) be a complete and separable metric space, $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ a metric dynamical system and $\varphi : \mathbb{R}_+ \times \Omega \times H \to H$ measurable with

- $\varphi(0, \omega) = \mathrm{id}$
- $\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)$ (cocycle property)
- $\varphi(t, \omega) : H \to H$ continuous.

Then φ is a random dynamical system (RDS).

e.g. $\varphi(t, \omega)x = X(t, 0, x)(\omega)$

Definition

- $K : \Omega \to 2^H$ is measurable if $\omega \mapsto d(x, K(\omega))$ is measurable for all $x \in H$, where $d(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$, $d(x, B) = d(\{x\}, B)$. Then K is called a random set.
- Let A, B be random sets. A is said to absorb B if $\mathbb{P}$-a.s. there exists an absorption time $t_B(\omega)$ such that for all $t \geq t_B(\omega)$

$$\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subseteq A(\omega).$$

- A is said to attract B if

$$d(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)) \xrightarrow[t \to \infty]{} 0, \ \mathbb{P}\text{-a.s.} \ .$$

Definition

A random attrator for an RDS φ is a random set A satisfying $\mathbb{P}$-a.s.

- (compactness): $A(\omega)$ is compact.
- (invariance): $\varphi(t,\omega)A(\omega) = A(\theta_t\omega)$ for all $t > 0$.
- (attraction): A attracts all deterministic bounded sets $B \subseteq H$, i.e.

$$d(\varphi(t,\theta_{-t}\omega)B, A(\omega)) \xrightarrow[t\to\infty]{} 0.$$

Existing results:

- No general results. Each equation treated individually.
- Semilinear equations. (Porous medium, p-Laplace not covered)
- Only special cases for the noise.

Remark: Porous medium equation
(Beyn, G., Lescot, Röckner; 2010)

$$dX_t = \Delta\Phi(X_t)dt + \sum_{j=1}^{m} \Phi_j d\beta_t^j,$$
$$X_t = 0, \text{ on } \partial\Lambda,$$

e.g. $\Phi(s) = |s|^{p-1}s$ and $\Phi_j \in W_0^{1,p+1}(\Lambda)$.

Variational framework for SPDE

Let V be a separable, reflexive Banach space, H a Hilbert space and $V \hookrightarrow H$ continuous and dense. This yields the Gelfand triple

$$V \subseteq H \cong H^* \subseteq V^*.$$

Consider:

$$dX_t = A(X_t)dt + dN_t.$$

In the sense

$$X(t,s,\omega,x) = x + \int_s^t A(X(r,s,\omega,x))dr + N_t(\omega) - N_s(\omega),$$

for all $t \geq s$ and all $\omega \in \Omega$.

Let $A : V \to V^*$ measurable and for some $C \in \mathbb{R}$, $\alpha \geq 2$, $\delta > 0$

(H1) (Hemicontinuity) The map $s \mapsto {}_{V^*}< A(v_1 + sv_2), v >_V$ is continuous on $\mathbb{R}$.

(H2) (Monotonicity)

$$2_{V^*}\langle A(v_1) - A(v_2), v_1 - v_2\rangle_V \leq C\|v_1 - v_2\|_H^2.$$

(H3) (Coercivity)

$$2_{V^*}\langle A(v), v\rangle_V + \delta\|v\|_V^\alpha \leq C + K\|v\|_H^2.$$

(H4) (Growth)

$$\|A(v)\|_{V^*} \leq C + C\|v\|_V^{\alpha-1}.$$

Generation of RDS

Let $Y(t,s,\omega,x)$ be the unique ω-wise solution of

$$Y_t = x - N_s(\omega) + \int_s^t A(Y_r + N_r(\omega))\mathrm{d}r, \;\; t \geq s.$$

Define

$$\begin{aligned} S(t,s,\omega)x &:= Y(t,s,x,\omega) + N_t(\omega), \\ \varphi(t,\omega)x &:= S(t,0,\omega)x = Y(t,0,x,\omega) + N_t(\omega). \end{aligned}$$

Note that S_t satisfies

$$S(t,s,\omega)x = x + \int_s^t A(S(r,s,\omega)x)\mathrm{d}r + N_t(\omega) - N_s(\omega),$$

for each fixed $\omega \in \Omega$ and all $t \geq s$.

Theorem

Let $A : V \to V^*$ satisfy (H1)-(H4), $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ be a metric dynamical system and $N_t : \Omega \to V$ be a process such that

- $(S1)$ *(Regularity) For each $\omega \in \Omega$, $N_.(\omega) \in L^\alpha_{loc}(\mathbb{R}; V)$.*
- $(S2)$ *For all $t, s \in \mathbb{R}$, $\omega \in \Omega$: $N_t(\omega) - N_s(\omega) = N_{t-s}(\theta_s\omega) - N_0(\theta_s\omega)$.*

Then

$$dX_t = A(X_t)dt + dN_t$$

generates a RDS φ (defined as above).

Single point attraction

Recall:

$(H2)$ (Monotonicity)

$$2{}_{V^*}\langle A(v_1) - A(v_2), v_1 - v_2 \rangle_V \leq C\|v_1 - v_2\|_H^2.$$

Stronger monotonicity:

$(H2')$ There exists a constant $\beta \geq 2$ and $\eta > 0$ such that

$$2{}_{V^*}\langle A(v_1) - A(v_2), v_1 - v_2 \rangle_V \leq -\eta\|v_1 - v_2\|_H^\beta, \ v_1, v_2 \in V.$$

Theorem

Suppose $(H1)$,$(H2')$, $(H3)$,$(H4)$ *and* $(S1)$-$(S2)$ *hold. Then* φ *has a compact random attractor* $\mathcal{A}(\omega)$ *consisting of a single point*

$$\mathcal{A}(\omega) = \{\eta_0(\omega)\}.$$

In particular there is a unique stationary solution $\eta_0(\omega)$ *and a unique invariant random measure* $\mu_\cdot \in \mathcal{P}_\Omega(H)$ *which is given by*

$$\mu_\omega = \delta_{\eta_0(\omega)}, \quad \mathbb{P}\text{-a.s. } .$$

Speed of convergence:

(i) If $\beta > 2$, then the speed of convergence is polynomial, more precisely

$$\|S(t,s,\omega)x - \eta_t(\omega)\|_H^2 \leq \left\{ \frac{\delta}{2}(\beta-2)(t-s) \right\}^{-\frac{2}{\beta-2}},$$

uniformly in $x \in H$.

(ii) If $\beta = 2$, then the speed of convergence is exponential, more precisely

$$\|S(t,s,\omega)x - \eta_t(\omega)\|_H^2 \leq 2\left(2\|x\|_H^2 + C\right) e^{-\delta(t-s)},$$

where $C > 0$ is a constant.

Existence of the random attractor

Proposition (cf. [CDF97, Theorem 3.11])

Let φ be an RDS and assume the existence of a compact random set K absorbing every deterministic bounded set $B \subseteq H$. Then there exists a random attractor A.

How to find a compact absorbing random set $K(\omega)$:

- Choose a subspace $S \subseteq H$ compact and

$$K(\omega) = B^S_{r(\omega)}(0) = \{x \in S| \; ||x||_S \le r(\omega)\}.$$

  Then $K(\omega) \subseteq H$ is compact. T.b.s.: $\forall t \ge T(\omega, B)$
  $\varphi(t, \theta_{-t}\omega)B \subseteq K(\omega)$
  $\Leftrightarrow \sup_{x \in B} ||\varphi(t, \theta_{-t}\omega)x||_S = \sup_{x \in B} ||S(0, -t, \omega)x||_S \le r(\omega)$

- Approximate $||\cdot||_S$ by weaker norms $\|\cdot\|_n \sim \|\cdot\|_H$ such that

$$||\cdot||_n \uparrow ||\cdot||_S$$

Condition $(H5)$**:**
Let $S \subseteq H$ be a compactly embedded subspace of H such that $V \subseteq S$. Let $T_n : H \to H$ be positive definite, self-adjoint operators and set

$$< x, y >_n := < x, T_n y >, \; \|x\|_n^2 = < x, x >_n \, .$$

e.g. $V = H_0^1(\Lambda) \cap L^p(\Lambda) \subseteq S = H_0^1(\Lambda) \subseteq H = L^2(\Lambda)$.
$T_n = \Delta(1 - \frac{1}{n}\Delta)^{-1} = n(I - (I - \frac{1}{n}\Delta)^{-1})$.

Further assume

1. $\|\cdot\|_n \sim \|\cdot\|_H$
2. For all $x \in S$,
$$\|x\|_n \uparrow \|x\|_S \ .$$
3. $T_n : V \to V$ are continuous
4. There exists a $C > 0$ such that
$$2_{V^*}\langle A(v), T_n v\rangle_V \leq C(\|v\|_n^2 + 1),\ \ v \in V. \tag{3.1}$$
5. There exists a measurable operator $T_\infty : \mathcal{D}(T_\infty) \subseteq V \to V$ such that $T_\infty N_\cdot(\omega) \in L^\alpha_{loc}(\mathbb{R}; V)$ and
$$\|T_n N_t\|_V \leq C\|T_\infty N_t\|_V,\ \ t \in \mathbb{R}, n \geq 1. \tag{3.2}$$

e.g. $T_n = \Delta(1 - \frac{1}{n}\Delta)^{-1} = n(I - (I - \frac{1}{n}\Delta)^{-1})$ and $T_\infty = \Delta$.

Theorem

Suppose $(H1)-(H5)$ *hold for* $\alpha = 2, K = 0$ *or* $\alpha > 2$ *and* $(S1)$-$(S2)$ *are satisfied. For the noise assume further*

$(S3)$ *(Subexponential growth) For* $\mathbb{P}$*-a.a.* $\omega \in \Omega$ *and* $|t| \to \infty$, $N_t(\omega)$ *is of subexponential growth, i.e.* $\|N_t(\omega)\|_V = o(e^{\lambda|t|})$ *for every* $\lambda > 0$.

$(S4)$ *(joint measurability):* $N : \mathbb{R} \times \Omega \to V$ *is* $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\Omega)$ *measurable.*

Then the RDS φ *associated to*

$$dX_t = A(X_t)dt + dN_t$$

has a compact random attractor.

Bounded absorption of $Y(t,s,x,\omega)$ at $t=-1$:

If $\alpha > 2$ or $\alpha = 2, K = 0$, then $\exists\lambda > 0$ and C such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Y_t\|_H^2 + \frac{\delta_0}{2}\|Y_t\|_V^\alpha \leq -\lambda\|Y_t\|_H^2 + f_t + C.$$

By Gronwall's Lemma, for $s \leq -1$,

$$\begin{aligned}\|Y_{-1}\|_H^2 &\leq e^{-\lambda(-1-s)}\|Y_s\|_H^2 + \int_s^{-1} e^{-\lambda(-1-r)}(f_r + C)\mathrm{d}r \\ &\leq 2e^{-\lambda(-1-s)}\|x\|_H^2 + 2\sup_{r\leq -1} e^{-\lambda(-1-r)}\|N_r(\omega)\|_H^2 \\ &\quad + \int_{-\infty}^{-1} e^{-\lambda(-1-r)}(f_r + C)\mathrm{d}r \leq r_1(\omega),\ \text{for } s \leq S(\omega,B).\end{aligned}$$

implies: $\frac{\delta_0}{2}\int_{-1}^0 \|Y_r\|_V^\alpha \mathrm{d}r \leq \|Y_{-1}\|_H^2 + \int_{-1}^0 (f_r + C)\mathrm{d}r.$ □

Compact absorption of $Y(t,s,x,\omega)$ at $t=0$:

Using Itô's formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Y_t\|_n^2 = 2\,{}_{V^*}\langle A(Y_t+N_t), T_nY_t\rangle_V$$
$$\le C(\|Y_t+N_t\|_n^2+1) - 2\,{}_{V^*}\langle A(Y_t+N_t), T_nN_t\rangle_V$$
$$\le C_1\left(\|Y_t\|_n^2+\|Y_t\|_V^\alpha\right)+g_t.$$

Gronwall's Lemma, for $s\le 0$,

$$\|Y_0\|_n^2 \le e^{-C_1 s}\|Y_s\|_n^2 + C_1\int_s^0 e^{-C_1 r}\|Y_r\|_V^\alpha \mathrm{d}r + \int_s^0 e^{-C_1 r} g_r \mathrm{d}r.$$

Integrating over $s\in[-1,0]$ and $n\to\infty$

$$\|Y_0\|_S^2 \le C_3\int_{-1}^0 e^{-C_1 r}\left(1+\|Y_r\|_V^\alpha\right)\mathrm{d}r + \int_{-1}^0 e^{-C_1 r} g_r \mathrm{d}r \le r_2(\omega).$$

□

Applications

1. Examples of noise N_t
2. Examples of SPDEs

The conditions on the noise

For generation of an RDS and singleton attractor:

1. $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ metric dynamical system
2. $N_t : \Omega \to V \subseteq H$
3. $(S1)$ (Regularity) $N_\cdot(\omega) \in L^\alpha_{loc}(\mathbb{R}; V)$, $\forall \omega$.
4. $(S2)$ $N_t(\omega) - N_s(\omega) = N_{t-s}(\theta_s \omega) - N_0(\theta_s \omega), \forall t, s, \omega$.

For the existence of a random attractor:

5. $(S3)$ (Subexponential growth) $\|N_t(\omega)\|_V = o(e^{\lambda|t|})$, $\forall \lambda > 0, \mathbb{P}$-a.a. ω, $|t| \to \infty$.
6. $(S4)$ (joint measurability): $N : \mathbb{R} \times \Omega \to V$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\Omega)$ measurable.

Lemma

Let $(N_t)_{t\in\mathbb{R}}$ *be a* V*-valued process with stationary increments and a.s. càdlàg paths. Then the canonical version* $\tilde{N}_t$ *satisfies* $(S1),(S2),(S4)$.

Note: $\Omega = D(\mathbb{R};V),\ \mathcal{F} = \mathcal{B}(V)^{\mathbb{R}}_{|\Omega},\ \theta_t(\omega) = \omega(t+\cdot)-\omega(t), \mathbb{P} = \mathcal{L}(N)$.

Lemma

Let N_t be a Lévy process with values in a separable Banach space V and with first moment, i.e. $\mathbf{E}||N_1||_V < \infty$. Then $\mathbb{P}$-a.s.

$$\frac{N_t}{t} \to \mathbf{E}[N_1], \text{ for } |t| \to \infty.$$

In particular N_t satisfies $(S3)$.

Lemma

Let N_t be a V-valued process such that $(S2)$ holds. Let there be constants $\gamma > 1$, $\alpha > 0$ and $C \in \mathbb{R}$ such that

$$\mathbf{E}[||N_t - N_s||_V^\gamma] \leq C|t-s|^{1+\alpha}, \text{ for all } t,s \in \mathbb{R}. \qquad (4.1)$$

Then there exists a constant $C = C(\omega) \in \mathbb{R}$ for which

$$||N_t(\omega)||_V \leq |t|^2 + C,$$

in particular N_t satisfies $(S3)$.

Example (Stochastic Porous Media equation)

For $r > 1$ we consider the triple

$$V := L^{r+1}(\Lambda) \subseteq H := (W_0^1(\Lambda))^* \subseteq V^*$$

and the stochastic porous media equation

$$\mathrm{d}X_t = \left(\Delta(|X_t|^{r-1}X_t) + \eta X_t\right)\mathrm{d}t + dN_t, \tag{4.2}$$

$\eta \in \mathbb{R}$, N_t satisfies $(S1) - (S4)$.
Then

1. The RDS φ associated with (4.2) has a compact random attractor.
2. If $\eta \leq 0$, then all assertions in Theorem 6 also hold for (4.2).

Let $\Lambda \subseteq \mathbb{R}^d$ be open, bounded, convex and smooth.

Example (Stochastic p-Laplace equation)

Consider the triple

$$W^{1,p}(\Lambda) \subseteq L^2(\Lambda) \subseteq (W^{1,p}(\Lambda))^*$$

and the stochastic p-Laplace equation

$$\mathrm{d}X_t = \left[\mathbf{div}(|\nabla X_t|^{p-2}\nabla X_t) - \eta_1 |X_t|^{\tilde{p}-2} X_t + \eta_2 X_t\right] \mathrm{d}t + dN_t, \qquad (4.3)$$

where $2 \le p < \infty, 1 \le \tilde{p} \le p$, $\eta_1 \ge 0$, $\eta_2 \in \mathbb{R}$ and N_t satisfies $(S1) - (S4)$. Then

1. The RDS φ associated with (4.3) has a compact random attractor.
2. If $\eta_2 \le 0$, then all assertions in Theorem 6 also hold for (4.3).

Hans Crauel, Arnaud Debussche, and Franco Flandoli, *Random attractors*, J. Dynam. Differential Equations **9** (1997), no. 2, 307–341.