

(Analytically) Strong Solutions for Stochastic Partial Differential Equations of Gradient Type

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Introduction and Motivation

- We consider SPDE of the form

$$dX_t = -\partial\varphi(X_t)dt + B_t(X_t)dW_t,$$

where $\varphi : H \rightarrow \bar{\mathbb{R}}$ is quasi-convex.

($\exists \lambda > 0$ such that $\varphi_\lambda := \varphi + \frac{\lambda}{2} \|\cdot\|_H^2$) is convex)

- $A := -\partial\varphi$, $\mathcal{D}(A) = \mathcal{D}(\partial\varphi) \subseteq H$

$$dX_t = A(X_t)dt + B_t(X_t)dW_t.$$

- Note: $\partial\varphi_\lambda(x) = \partial\varphi(x) + \lambda x$.

Example (Stochastic Porous Medium Equation)

Consider

$$dX_t = \Delta(|X_t|^{p-1}X_t)dt + B_t(X_t)dt, \quad (p \geq 1),$$

on $\mathcal{O} \subseteq \mathbb{R}^d$ open, bounded or $\mathcal{O} = \mathbb{R}^d$. With $H = (H_0^1(\mathcal{O}))^*$,

$$\varphi(v) = \frac{1}{p+1} \|v\|_{L^{p+1}(\mathcal{O})}^{p+1},$$

this is of the form

$$dX_t = -\partial\varphi(X_t)dt + B_t(X_t)dW_t.$$

Strong Solutions

- SPDE have less (spatial) regularity than PDE.
- Solution: Consider mild or variational solutions
 - mild approach: A generator, $dX_t = (AX_t + F(X_t))dt + B(X_t)dW_t$,

$$X_t = e^{At}X_0 + \int_0^t e^{A(t-s)}F(X_s)ds + \int_0^t e^{A(t-s)}B(X_s)dW_s$$

with X predictable in H ($X \in \mathcal{D}(F) \cap \mathcal{D}(B)$).

- variational approach: Gelfand triple $V \subseteq H \subseteq V^*$, extend A to $A : V \rightarrow V^*$, variational solution $X \in V$ a.s. (e.g. $\mathcal{D}(A) = \{h \in L^{p+1} \mid \Phi(h) \in H_0^1\}$, $V = L^{p+1}$).

Strong Solutions

Definition (Strong Solution)

Let $X_0 \in L^2(\Omega, \mathcal{F}_0; H)$. $X \in L^2(\Omega; C([0, T]; H))$ is a strong solution if X is \mathcal{F}_t -adapted, continuous in H

- $\partial\varphi(X) \in L^2([0, T] \times \Omega; H)$
- \mathbb{P} -a.s.

$$X_t = X_0 - \int_0^t \partial\varphi(X_r) dr + \int_0^t B_r(X_r) dW_r, \quad \forall t \in [0, T].$$

If $A = -\partial\varphi$ and B sufficiently regular we prove the unique existence of strong solutions.

Regularization (Deterministic Case)

Linear Equations: $dX_t = AX_t dt$

- If A analytic, $X_0 \in H$: $X_t \in \mathcal{D}(A)$ for a.e. $t > 0$ and X_t is a strong solution, i.e.

$$\frac{dX}{dt} = AX_t, \text{ a.e.}$$

- Note: If $A = -\partial\varphi$ then A is analytic.

Non-linear Equations: $dX_t = A(X_t)dt$,

- If $A = -\partial\varphi$, $X_0 \in \overline{\mathcal{D}(\varphi)}$ then $t^{\frac{1}{2}}\partial\varphi(X_t) \in L^2([0, T]; H)$, $\varphi(X) \in L^1([0, T])$ and

$$\frac{dX}{dt}(t) + \partial\varphi(X_t) = f_t, \text{ a.e. } t \in (0, T).$$

Regularization for SPDE

If $A = -\partial\varphi$, B sufficiently smooth, $X_0 \in L^2(\mathcal{O}; H)$ then

$$\begin{aligned}\varphi(X) + \|X\|_H^2 &\in L^1([0, T] \times \Omega), \\ t^{\frac{1}{2}}\partial\varphi(X_t) &\in L^2([0, T] \times \Omega; H)\end{aligned}$$

and X is the unique strong solution on each interval $(\delta, T]$, $\delta > 0$ (in particular $X \in \mathcal{D}(A)$ a.s.).

Unified Framework

- Mild approach: Restricted to semilinear equations (e.g. not SPME)
- Variational approach: Not applicable to RDE with high-order growth of the reaction term
($dX_t = (\Delta X_t + X_t^p)dt + B(X_t)dW_t$, $p \geq 1$ arbitrary).
- We present a framework which contains both types of equations (also applicable on not necessarily bounded domains and with various boundary conditions).

Method of Proof

Regularization in Deterministic Case

- Proof based on chain rule for subgradients: Let $X \in W^{1,2}([0, T]; H)$, $X_t \in \mathcal{D}(\partial\varphi)$ for a.e. $t \in [0, T]$, then

$$\varphi(X_t) = \varphi(X_0) + \int_0^t (g_\tau, \frac{dX}{dt}(\tau))_H d\tau,$$

with $g_\tau \in \partial\varphi(X_\tau)$ a.e.

- Proof via Moreau-Yosida approximation φ_λ .
Note: $\partial\varphi_\lambda = (\partial\varphi)_\lambda \rightarrow (\partial\varphi)^0$ on $\mathcal{D}(\partial\varphi)$.
- Problem in the stochastic case:

$$\begin{aligned} \varphi_\lambda(X_t) &= \varphi_\lambda(X_0) + \int_0^t (\partial\varphi_\lambda(X_\tau), dX_\tau)_H \\ &\quad + \int_0^t \text{Tr}[D^2\varphi_\lambda(X_\tau)B(X_\tau)B(X_\tau)^*]d\tau. \end{aligned}$$

Would require uniform control on $D^2\varphi_\lambda$.

Standard Galerkin approach

- Let $P_n : H \rightarrow H_n$ orthogonal projection onto H_n (w.r.t. $\|\cdot\|_H$).

$$X_t^n = X_0^n - \int_0^t P_n \partial\varphi(X_\tau^n) d\tau + \int_0^t P_n B(X_\tau^n) dW_\tau^n.$$

By Itô's formula:

$$\begin{aligned} \varphi(X_t^n) &= \varphi(X_0^n) + \int_0^t (\partial\varphi(X_\tau^n), dX_\tau^n)_H \\ &\quad + \int_0^t \text{Tr}[D^2\varphi(X_\tau^n) P_n B(X_\tau^n) P_n B(X_\tau^n)^*] d\tau. \end{aligned}$$

- Problem: $\varphi(P_n h) \not\leq C\varphi(h)$, i.e. the Galerkin-approximation based on $\|\cdot\|_H$ is not compatible with the "geometry" of φ .

Main Idea

- Galerkin approximation weighted by the "distance" given by φ .
- Recall: $P_n : H \rightarrow H_n$ (orth. projection) is the $\|\cdot\|_H$ -best-approximation, i.e.

$$\|h - P_n h\|_H = \inf_{g \in H_n} \|h - g\|_H.$$

Hence, $\|P_n h\|_H \leq 2\|h\|_H$.

- Idea: Consider $\mathcal{P}_n : H \rightarrow H_n$ the φ -best-approximation, i.e.

$$\varphi(h - \mathcal{P}_n h) = \inf_{g \in H_n} \varphi(h - g).$$

Then

$$\varphi(\mathcal{P}_n h) \leq C\varphi(h)$$

$$\varphi(\mathcal{P}_n h) \rightarrow \varphi(h).$$

Weighted Galerkin Approximation

- Galerkin approximation:

$$X_t^n = \mathcal{P}_n X_0 - \int_0^t \mathcal{P}_n \partial \varphi(X_\tau^n) d\tau + \int_0^t \mathcal{P}_n B(X_\tau^n) dW_\tau^n.$$

- Note: Loose joint monotonicity of $A = -\partial \varphi$, B , i.e.

$$(P^n A(v) - P^n A(w), v - w)_H + \|\mathcal{P}_n B(v) - \mathcal{P}_n B(w)\|_H^2 \not\leq C \|v - w\|_H^2.$$

Solution: Freeze noise (additive case) then use fixed point theorem.

Setup and Main Results

General Setup

Let H be a separable Hilbert space, S a Banach space, V a reflexive Banach space satisfying

- $V \subseteq S, V \subseteq H \subseteq V^*$.
- $\|\cdot\|_V \leq C(\|\cdot\|_S + \|\cdot\|_H)$.

(e.g. $H = (H_0^1)^*$, $S = L^{p+1}$, " $V = S \cap H$ ")

Let U be a separable Hilbert space, $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ a complete, normal, filtered probability space, W_t a cylindrical Brownian motion on U and $B : [0, T] \times \Omega \times H \rightarrow L_2(U, H)$ progressively measurable.

Let $\{e_k \in V \mid k \in \mathbb{N}\}$ be an orthonormal basis of H , such that $V_0 = \text{span}\{e_1, \dots\}$ is dense in V . Define $H_n := \text{span}\{e_1, \dots, e_n\}$.

Assumptions on the drift

- (A1)
- $\varphi : S \rightarrow \mathbb{R}_+$ convex, continuous
 - subhomogeneous
(i.e. $\exists C > 0$ such that $\varphi(2x) \leq C\varphi(x)$ for all $x \in S$),
 - bounded level-sets,
 - $\varphi(v_n) \rightarrow 0$ implies $v_n \rightarrow 0$ in S (e.g. $\varphi = \frac{\|\cdot\|_{p+1}^{p+1}}{p+1}$, $S = L^{p+1}$).
- (A2)
- $\varphi : V \rightarrow \mathbb{R}_+$ Gateaux differentiable, $A := -D\varphi : V \rightarrow V^*$ hemicontinuous, (e.g. $D\varphi(v)h = \int_{\mathcal{O}} |v|^{p-1}vh \, d\xi$)
 - $\varphi \in C^2(H_n)$, $\forall n \in \mathbb{N}$, with constants $1 = p_0 \leq p_2 \leq \dots \leq p_N$

$$\sum_{k=1}^{\infty} D^2\varphi(x)(w_k, w_k) \leq C \left(1 + \varphi(x) + \sum_{i=0}^N \left(\sum_{k=1}^{\infty} \varphi_1(w_k)^{\frac{1}{p_i}} \right)^{p_i} \right),$$

for each sequence $w_k \in H_n$, all $x \in H_n$.

(e.g. $D^2\varphi(x)(w_k, w_k) = p \int_{\mathcal{O}} |x|^{p-1} |w_k|^2 \, d\xi$).

(A3) $\varphi : H \rightarrow \bar{\mathbb{R}}$ quasi-convex (i.e. $\exists \lambda \geq 0$ such that φ_λ is convex).

(A4) (Weak coercivity): For all $v \in V$,

$$2 \nu^* \langle -D\varphi(v), v \rangle_V \leq C(1 + \|v\|_H^2).$$

(e.g. $2 \nu^* \langle -D\varphi(v), v \rangle_V = -\int_{\mathcal{O}} |v|^{p+1} = -(p+1)\varphi(v) \leq 0$).

Assumptions on the noise

(A5) (Lipschitz noise):

$$\|B_t(v) - B_t(w)\|_{L_2(U,H)}^2 \leq c\|v - w\|_H^2, \quad \forall v, w \in V.$$

(A6) (Regularity of the noise): Exists \tilde{e}_k orthonormal basis of U ,

$$\begin{aligned} \|B_t(v)\|_{L_2, \tilde{\varphi}_1, (p_i)} &:= \sum_{i=0}^N \left(\sum_{k=1}^{\infty} \varphi_1(B_t(v)(\tilde{e}_k))^{\frac{1}{p_i}} \right)^{p_i} \\ &\leq C(f_t + \varphi(v) + \|v\|_H^2), \end{aligned}$$

p_i as in (A2), $f_t \in L^1([0, T] \times \Omega)$ is \mathcal{F}_t -adapted.

Theorem (Strong Solution)

Assume (A1)-(A6), $X_0 \in L^2(\Omega; H)$ and $\mathbb{E}(\varphi(X_0)) < \infty$. There is a unique strong solution X with

$$\varphi(X) + \|X\|_H^2 \in L^\infty([0, T]; L^1(\Omega))$$

(recall $\partial\varphi(X) \in L^2([0, T] \times \Omega; H)$).

Definition (Limit Solution)

Let $X_0 \in L^2(\Omega; H)$. $X \in L^2(\Omega; C([0, T]; H))$ is a limit solution if X is \mathcal{F}_t -adapted, $X(0) = X_0$ and there exists a sequence X^n of strong solutions such that $X^n \rightarrow X$ in $L^2(\Omega; C([0, T]; H))$.

Theorem (Limit Solution)

Assume (A1)-(A6), $X_0 \in L^2(\Omega; H)$. Then there exists a unique limit solution X .

Definition

Let $X_0 \in L^2(\Omega; H)$. $X \in L^2(\Omega; C([0, T]; H))$ is a strong solution on $(0, T]$ if X is \mathcal{F}_t -adapted, $X(0) = X_0$, with

- $\partial\varphi(X) \in L^2([\delta, T] \times \Omega; H)$,
- \mathbb{P} -a.s.

$$X_t = X_\delta - \int_\delta^t \partial\varphi(X_r) dr + \int_\delta^t B_r(X_r) dW_r, \quad \forall t \in [\delta, T],$$

for all $0 < \delta < T$.

(A4') There exist constants $C_1 > 0$, $C_2 \in \mathbb{R}$ such that

$$2 \nu^* \langle -D\varphi(\nu), \nu \rangle_V \leq C_2(1 + \|\nu\|_H^2) - C_1\varphi(\nu),$$

for all $\nu \in V$. (e.g. $2 \nu^* \langle -D\varphi(\nu), \nu \rangle_V = -(p+1)\varphi(\nu) \leq 0$).

Theorem

Assume (A1)-(A6), (A4'). Let $X_0 \in L^2(\Omega; H)$ and X be the corresponding limit solution. Then

$$\begin{aligned} \varphi(X) + \|X\|_H^2 &\in L^1([0, T] \times \Omega), \\ t^{\frac{1}{2}} \partial\varphi(X_t) &\in L^2([0, T] \times \Omega; H) \end{aligned}$$

and X is the unique strong solution on $(0, T]$.

Applications

Stochastic Porous Medium Equation

Consider

$$dX_t = \Delta(|X_t|^{p-1}X_t)dt + B_t(X_t)dW_t,$$

on a bounded domain $\mathcal{O} \subseteq \mathbb{R}^d$ with Dirichlet boundary conditions. Let $H := (H_0^1(\mathcal{O}))^*$, $V = S := L^{p+1}(\mathcal{O})$, and $\varphi(x) := \frac{1}{p+1} \|x\|_{p+1}^{p+1}$.

Theorem (Standard SPME)

Let $X_0 \in L^2(\Omega; H)$, B satisfy (A5), (A6). There exists a unique strong solution X on $(0, T]$ with $X(0) = X_0$ and

$$\mathbb{E} \int_0^T \|X_t\|_{p+1}^{p+1} + t \| |X_t|^{p-1} X_t \|_{H_0^1}^2 dt < \infty.$$

The variational solution obtained in [RRW07, Theorem 3.9] coincides with this strong solution.

If additionally $\mathbb{E} \|X_0\|_{p+1}^{p+1} < \infty$, then X is a strong solution with

$$\sup_{t \in [0, T]} \mathbb{E} \|X_t\|_{p+1}^{p+1} + \mathbb{E} \int_0^T \| |X_t|^{p-1} X_t \|_{H_0^1}^2 dt < \infty.$$

Remark (Generalized SPME)

$$dX_t = L\Phi(X_t)dt + B_t(X_t)dW_t,$$

L transient generator.

Stochastic Reaction Diffusion Equation

Consider

$$dX_t = (\Delta X_t + f(X_t)) dt + B_t(X_t) dW_t,$$

on a bounded domain $\mathcal{O} \subseteq \mathbb{R}^d$ with Dirichlet boundary conditions, f being a polynomial of odd degree N , with negative leading coefficient. Let $H = L^2(\mathcal{O})$, $S = V = H_0^1(\mathcal{O}) \cap L^{N+1}(\mathcal{O})$ and

$$\varphi(v) := \frac{1}{2} \|v\|_{H_0^1}^2 - \int_{\mathcal{O}} F(v) d\xi,$$

where $F' = f$.

Theorem (Standard SRDE)

Let $X_0 \in L^2(\Omega, \mathcal{F}_0; H)$ and B satisfy (A5), (A6). There exists a unique strong solution X on $(0, T]$ with $X(0) = X_0$ and

$$\mathbb{E} \int_0^T \left(\|X_t\|_{H_0^1}^2 + \|X_t\|_{N+1}^{N+1} \right) + t \left(\|X_t\|_{H^2}^2 + \|X_t\|_{2N}^{2N} \right) dt < \infty.$$

If $\mathbb{E} \left(\|X_0\|_{H_0^1}^2 + \|X_0\|_{N+1}^{N+1} \right) < \infty$, then X_t is the unique strong solution with

$$\sup_{t \in [0, T]} \mathbb{E} \left(\|X_t\|_{H_0^1}^2 + \|X_t\|_{N+1}^{N+1} \right) + \mathbb{E} \int_0^T \left(\|X_t\|_{H^2}^2 + \|X_t\|_{2N}^{2N} \right) dt < \infty.$$

Remark (Generalized SRDE)

$$dX_t = \left(LX_t + \sum_{i=1}^N f_i(X_t) \right) dt + B_t(X_t) dW_t,$$

L non-negative, self-adjoint on $L^2(m)$.

Stochastic p -Laplace Equation

Consider

$$dX_t = (\operatorname{div}(|\nabla X_t|^{p-2} \nabla X_t) + f(X_t)) dt + B_t(X_t) dW_t,$$

on a bounded domain $\mathcal{O} \subseteq \mathbb{R}^d$, f being a polynomial of odd degree N , with negative leading coefficient. Let $H = L^2(\mathcal{O})$, $V = S = W_0^{1,p}(\mathcal{O}) \cap L^{N+1}(\mathcal{O})$ and

$$\varphi(v) := \frac{1}{p} \int_{\mathcal{O}} |\nabla v|^p d\xi - \int_{\mathcal{O}} F(v) d\xi.$$

Theorem

Let $X_0 \in L^2(\Omega, \mathcal{F}_0; H)$, B satisfy (A5), (A6). There exists a unique strong solution X on $(0, T]$ with $X(0) = X_0$ and

$$\mathbb{E} \int_0^T \left(\|X_t\|_{W_0^{1,p}(\mathcal{O})}^p + \|X_t\|_{N+1}^{N+1} \right) + t \left(\|\operatorname{div}(|\nabla X_t|^{p-2} \nabla X_t)\|_2^2 + \|X_t\|_{2N}^{2N} \right) dt < \infty.$$

If $\mathbb{E} \left(\|X_0\|_{W_0^{1,p}(\mathcal{O})}^p + \|X_0\|_{N+1}^{N+1} \right) < \infty$, then X_t is the unique strong solution with

$$\sup_{t \in [0, T]} \mathbb{E} \left(\|X_t\|_{W_0^{1,p}(\mathcal{O})}^p + \|X_t\|_{N+1}^{N+1} \right) + \mathbb{E} \int_0^T \|\operatorname{div}(|\nabla X_t|^{p-2} \nabla X_t)\|_2^2 + \|X_t\|_{2N}^{2N} dt < \infty.$$

Remark

$dX_t = (\operatorname{div}(\Phi(\nabla X_t)) + \sum_{i=1}^n f_i(X_t)) dt + B_t(X_t) dW_t$,
 $\Phi = \nabla \Psi$, for some convex function Ψ with polynomial coercivity and growth.

Existing Results (not exhaustive)

Linear SPDE:

[Fla90, Fla95, Roz90, RS81, Kry94, DPZ92, BvN03, BvNVW08].

Semilinear SPDE:

[JR10, DPZ92].

Quasilinear SPDE:

[Liu10, RW08, Kim06, BGLR11].



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