Random Attractor for Stochastic Porous Media Equations

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3. RDS given by Stochastic Porous Media Equations
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Why study long-time behaviour?

At least two crucial aims:

- Understand chaotic behaviour, turbulence
- Reduction of complexity (ergodicity)

Chaotic behaviour

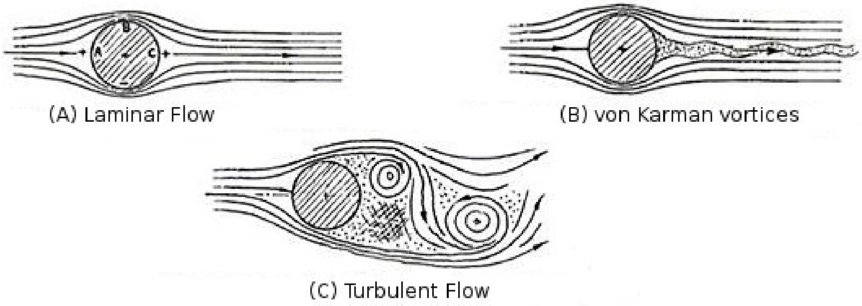


Figure: Transition to chaotic behaviour

Reduction of complexity

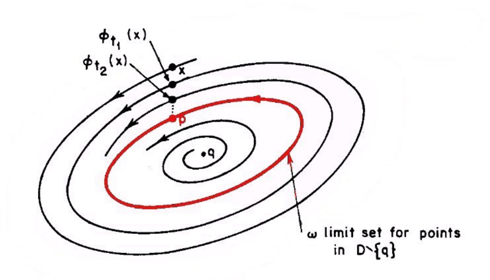
'Fully developed' dynamics (i.e. that have been run a long time) have reduced complexity. There exists a 'small' set $A \subseteq H$ (H the state space) such that

- A is invariant under the flow
- Each trajectory can be approximated arbitrarily well by one lying in A (for large times)
- A is minimal.

Sets with these properties are called attractors.

Examples (Reduction of complexity):

- 2d-attractor



- 2d Navier-Stokes equation:

$$\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f.$$

Consider dynamics $u_t \in H \approx L^2(\mathcal{O}; \mathbb{R}^2)$.
Key result: There exists an attractor A of finite (Hausdorff) dimension. Even have an embedding $A \hookrightarrow \mathbb{R}^d$.

Basics on Random Dynamical Systems (RDS)

Definition (1.1)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta_t : \Omega \to \Omega$, $t \in \mathbb{R}$, be a family of maps. Then $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ is said to be a **metric dynamical system** if

- $\theta : \mathbb{R} \times \Omega \to \Omega$ is measurable
- $\theta_0 = id$, $\theta_{t+s} = \theta_t \circ \theta_s$
- $(\theta_t)_* \mathbb{P} = \mathbb{P}$

e.g. $\Omega = C_0(\mathbb{R}; \mathbb{R})$, $\mathbb{P} =$ Wiener measure, $\theta_t(\omega) = \omega(t + \cdot) - \omega(t)$ ("Wiener shift").

Basics on Random Dynamical Systems (RDS)

Definition (1.2)

Let (H, d) be a complete and separable metric space, $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ a metric dynamical system and $\varphi : \mathbb{R}_+ \times \Omega \times H \to H$ measurable with

- $\varphi(0, \omega) = \mathrm{id}$
- $\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)$ (cocycle property)
- $\varphi(t, \omega) : H \to H$ continuous.

Then φ is called a **random dynamical system** (RDS).

Definition (1.3)

- $K : \Omega \to 2^H$ is called **measurable** if $\omega \mapsto d(x, K(\omega))$ is measurable for all $x \in H$, where d is the Hausdorff semidistance. K is also called a **random set**.

Basics on Random Dynamical Systems (RDS)

Definition (1.3)

- Let A, B be random sets. A is said to **absorb** B if for $\mathbb{P}$-a.e. $\omega \in \Omega$ there exists an **absorption time** $t_B(\omega)$ such that for all $t \geq t_B(\omega)$

$$\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subseteq A(\omega).$$

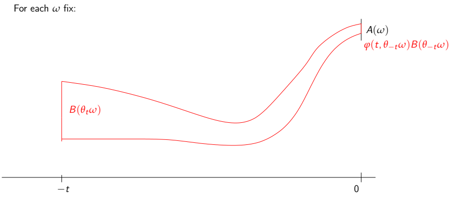


Figure: pullback absorption

Basics on Random Dynamical Systems (RDS)

Definition (1.3)

- A is said to **attract** B if

$$d(\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)) \xrightarrow[t\to\infty]{} 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

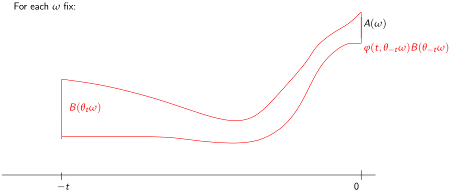


Figure: pullback attraction

Basics on Random Dynamical Systems (RDS)

Definition (1.4)

A **random attrator** for an RDS φ is a random set A satisfying $\mathbb{P}$-a.s.

- (compactness): $A(\omega)$ is compact.
- (invariance): $\varphi(t,\omega)A(\omega) = A(\theta_t\omega)$ for all $t > 0$.
- (attraction): A attracts all deterministic bounded sets $B \subseteq H$, i.e.

$$d(\varphi(t,\theta_{-t}\omega)B, A(\omega)) \xrightarrow[t\to\infty]{} 0.$$

RDS given by Stochastic Porous Media Equations

Consider a stochastic porous media equation over a bounded open set $\Lambda \subset \mathbb{R}^d$

$$dX_t = \Delta(\Phi(X_t))\, dt + Q dW_t,\ \ t \geq s, \qquad \text{(SPME)}$$

where $t, s \in \mathbb{R}$ and for $m \in \mathbb{N}$

$$QW_t = \sum_{j=1}^{m} \varphi_j \beta_j,$$

$\varphi_1, \ldots, \varphi_m \in C_0^1(\Lambda)$, $\beta_1, \ldots, \beta_m$ independent $\mathbb{R}$-valued Brownian motions on canonical Wiener space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ (i.e. $\Omega = C(\mathbb{R}; \mathbb{R}^m)$), and

RDS given by Stochastic Porous Media Equations

- $\Phi : \mathbb{R} \to \mathbb{R}$ continuous, $\Phi(0) = 0$ (for simplicity)
- Φ monotone
- $\exists p \geq 1$, $a > 0$, $c \geq 0$ such that

$$\Phi(s)s \geq a|s|^{p+1} - c \quad \forall s \in \mathbb{R} \text{ ("coercive")};$$

- $\Phi(s) \leq \text{const.}(|s|^p + 1) \quad \forall s \in \mathbb{R}$ ("polynomial growth").

RDS given by Stochastic Porous Media Equations

Consider the triple

$$V := L^{p+1}(\Lambda) \subset H \subset V^*,$$

where $H := (H_0^1(\Lambda))^*$. By "shifting the noise" (SPME) reduces to deterministic porous media equation (PME) with time and ω-dependent coefficients, i.e. change of variables

$$X_t \rightsquigarrow X_t - QW_t =: Y_t$$

yields (in integral form) for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$Y_t(\omega) = x - QW_s(\omega) + \int_s^t A_\omega(r, Y_r(\omega))dr, \quad t \geq s, \quad \text{(PME)}$$

where $x \in H$ and

$$A_\omega(r, \cdot) := \Delta\Phi(\cdot + QW_r(\omega)).$$

RDS given by Stochastic Porous Media Equations

By standard variational methods (PME) has a unique solution for each fix $\omega \in \Omega$:

$$Y(\cdot, s, x, \omega) \in L^{p+1}_{\text{loc}}([s, \infty); V) \cap C([s, \infty), H).$$

Then reversing "shift-transformation"

$$X_t(\omega) := Y(t, s, x, \omega) + QW_t(\omega) \quad , \quad t \geq s; \; t, s \in \mathbb{R},$$

is stochastic process which solves

$$X_t = x + \int_s^t \Delta\Phi(X_r) dr + QW_t - QW_s. \qquad \text{(SPME)}$$

Then

$$\varphi(t, \omega)x := X(t, 0, \omega)x, \quad t \geq 0,$$

defines an RDS.

RDS given by Stochastic Porous Media Equations

Hypothesis (Assumptions 1)

Assume $\varphi_j \in H_0^{2,p+1}(\Lambda)$*, thus* $QW_t \in H_0^{2,p+1}(\Lambda)$*. Let further* $\zeta : \mathbb{R} \to \mathbb{R}$*,* $\zeta(0) = 0$ *be a function such that*

- ζ*-Weak monotonicity: For all* $t, s \in \mathbb{R}$

$$(\Phi(t) - \Phi(s))(t - s) \geq (\zeta(t) - \zeta(s))^2.$$

- ζ*-Coercivity: For* p, a, c *as in (A2) and for all* $s \in \mathbb{R}$

$$\Phi(s)s \geq \zeta(s)^2 \geq a|s|^{p+1} - c.$$

RDS given by Stochastic Porous Media Equations

Hypothesis (Assumptions 2)

Let $\varphi_j \in H_0^{1,p+1}(\Lambda)$. *Assume some further properties of* $\Phi \in C^1(\mathbb{R})$.

Main Results

Theorem (3.1)

Under the above conditions on (QW and) Φ the RDS given by (SPME) as defined above has a random attractor.

Main Results

Theorem (3.2)

Assume that (QW_t) is a V-valued Wiener process and that $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous satisfying:

- $\exists p > 1, \eta > 0$ *such that*

$$(t-s)(\Phi(t) - \Phi(s)) \geq \eta |t-s|^{p+1} \quad \forall s, t \in \mathbb{R}.$$
$$|\Phi(s)| \leq const.(1 + |s|^p) \quad \forall s \in \mathbb{R}.$$

(Example: $\Phi(s) = s|s|^{p-1}$.)
Then the random attractor given by (SPME) as defined above is given by

$$A(\omega) := \lim_{s \to -\infty} X(0, s, \omega)x,$$

(which is independent of x!), i.e. is a random point.

Idea of Proof

Proposition (1.6 [Crauel/Flandoli: PTRF 1994])

Let φ be an RDS and assume the existence of a compact random set K absorbing every deterministic bounded set $B \subseteq H$. Then there exists a random attractor A.

Idea of Proof

Steps

1. Bounded absorption: Bound for the H-norm

$$\frac{d}{dt}||Y_t||_H^2 \leq -\alpha||X_t||_{L^2}^2 + p_1(t), \qquad \circledast$$

for some $\alpha > 0$ and some polynomial p_1 with random coefficients.

2. Compact absorption: Bound for the stonger L^2-norm

$$\frac{d}{dt}||Y_t||_{L^2}^2 \leq -\alpha||Y_t||_{L^2}^2 + p_2(t), \qquad \circledast\circledast$$

for some $\alpha > 0$ and some polynomial p_2 with random coefficients.

Idea of Proof

Remark The bound

$$\frac{d}{dt}||Y_t||_{L^2}^2 \leq -\alpha||Y_t||_{L^2}^2 + p_2(t), \qquad (*)(*)$$

in particular implies that $t \to Y_t(\omega)$ is right continuous in $L^2(\Lambda)$.
Remark Higher regularity:

$$\Phi(X(\cdot, s, \omega)x) \in L_{loc}^{\frac{p+1}{p}}([s, \infty); H_0^{1, \frac{p+1}{p}}),$$

for $x \in L^2(\Lambda)$, $s \in \mathbb{R}$ and $\omega \in \Omega$

Idea of Proof

The first bound ⊛ follows by the usual coercivity of $\Delta\Phi : V \to V^*$:
Recall

$$Y_t = x - QW_s(\omega) + \int_s^t \Delta\Phi(Y_r + QW_r(\omega))dr, \quad t \geq s$$

which in differential form is

$$\frac{d}{dr} Y_r = \Delta\Phi(Y_r + QW_r(\omega)) = \Delta\Phi(X_r), \quad t \geq s.$$

For dr–a.e. $r \in [s, \infty)$

$$\begin{aligned}
\frac{d}{dr} \|Y_r\|_H^2 &= -2\langle Y_r, \Phi(X_r)\rangle \\
&= -2\langle X_r - QW_r, \Phi(X_r)\rangle \\
&= -2\langle X_r, \Phi(X_r)\rangle + 2\langle QW_r, \Phi(X_r)\rangle \\
&\leq -2a \int_\Lambda |X_r|^{p+1} d\xi + 2 \int_\Lambda \left(|QW_r \Phi(X_r)| + c\right) d\xi,
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $L^2(\Lambda)$.

Idea of Proof

Formal derivation of ⊛⊛ for $\Phi(s) = s^p$ (p odd)
Since

$$\frac{d}{dr} Y_r = \Delta\Phi(Y_r + QW_r(\omega)) = \Delta(X_r)^p, \quad t \geq s.$$

we have

$$\begin{aligned}\frac{d}{dr}\|Y_r\|_{L^2}^2 &= 2\langle Y_r, \Delta(X_r)^p\rangle \\ &= 2\langle X_r, \Delta(X_r)^p\rangle - 2\langle QW_r(\omega), \Delta(X_r)^p\rangle\end{aligned}$$

Idea of Proof

For the first (good) term:

$$\begin{aligned}
2\langle X_r, \Delta(X_r)^p\rangle &= -2\langle \nabla X_r, \nabla(X_r)^p\rangle \\
&= -2p\langle \nabla X_r, \nabla X_r (X_r)^{p-1}\rangle \\
&= -2p\langle (X_r)^{\frac{p-1}{2}} \nabla X_r, \nabla X_r (X_r)^{\frac{p-1}{2}}\rangle \\
&= -2\frac{4p}{(p+1)^2}\langle \nabla X_r^{\frac{p+1}{2}}, \nabla X_r^{\frac{p+1}{2}}\rangle \\
&= -2\frac{4p}{(p+1)^2}||\underbrace{X_r^{\frac{p+1}{2}}}_{\in H_0^1}||_{H_0^1}^2
\end{aligned}$$

Idea of Proof

For the second (bad) term:

$$-2\langle QW_r(\omega), \Delta (X_r)^p\rangle$$

$$\begin{aligned}
&= 2\langle \nabla QW_r(\omega), \nabla \overbrace{(X_r)^p}^{\notin H_0^1}\rangle \\
&= 2p\langle \nabla QW_r(\omega), (X_r)^{p-1}\nabla \underbrace{X_r}_{\notin H_0^1}\rangle \\
&= 2p\langle (X_r)^{\frac{p-1}{2}}\nabla QW_r(\omega), (X_r)^{\frac{p-1}{2}}\nabla X_r\rangle \\
&= 2\frac{2p}{p+1}\langle (X_r)^{\frac{p-1}{2}}\nabla QW_r(\omega), \nabla \underbrace{(X_r)^{\frac{p+1}{2}}}_{\in H_0^1}\rangle \\
&\le \varepsilon||\nabla (X_r)^{\frac{p+1}{2}}||_{L^2}^2 + \varepsilon||X_r^{\frac{p+1}{2}}||_{L^2}^2 + C_\varepsilon||\nabla QW_r(\omega)||_{L^{p+1}}^{p+1} \\
&\le 2\varepsilon||\nabla (X_r)^{\frac{p+1}{2}}||_{L^2}^2 + C_\varepsilon||\nabla QW_r(\omega)||_{L^{p+1}}^{p+1}
\end{aligned}$$

Idea of Proof

Hence

$$\frac{d}{dr}\|Y_r\|_{L^2}^2 = 2\langle Y_r, \Delta(X_r)^p\rangle$$
$$= 2\langle X_r, \Delta(X_r)^p\rangle - 2\langle QW_r(\omega), \Delta(X_r)^p\rangle$$
$$\leq \underbrace{\left(2\varepsilon - 2\frac{4p}{(p+1)^2}\right)}_{<0} ||X_r^{\frac{p+1}{2}}||_{H_0^1}^2 + C_\varepsilon||\nabla QW_r(\omega)||_{L^{p+1}}^{p+1}.$$

Idea of Proof

Rigorous proof of

$$\frac{d}{dt}||Y_t||_{L^2}^2 \leq -\alpha||Y_t||_{L^2}^2 + p_2(t): \qquad (*)(*)$$

Consider H with inner products

$$\text{``}\left\langle (1-\epsilon\Delta)^{-1}x, y \right\rangle_{L^2}\text{''} =: \langle x, y\rangle_{H_\epsilon}, \quad \epsilon > 0,$$

and norm $||\cdot||_{H_\epsilon} = \sqrt{\langle \cdot, \cdot \rangle_{H_\epsilon}}$. Note $||\cdot||_{H_\epsilon} \uparrow ||\cdot||_{L^2}$ as $\epsilon \downarrow 0$.
By the usual chain rule:

$$\begin{aligned} ||Y_t||_{H_\epsilon}^2 = {} & ||Y_0||_{H_\epsilon}^2 \\ & + \frac{2}{\epsilon}\int_0^t \left\langle \Phi(X_s), (1-\epsilon\Delta)^{-1}X_s - X_s \right\rangle_{L^2} ds \\ & - 2\int_0^t \left\langle \Phi(X_s), \Delta(1-\epsilon\Delta)^{-1}(QW_s) \right\rangle_{L^2} ds \end{aligned}$$

Idea of Proof

Recall for $x \in L^2(\Lambda)$

$$(1-\epsilon\Delta)^{-1}x(\xi) = \int_\Lambda x(\tilde{\xi}) \underbrace{g_\epsilon(\xi,\tilde{\xi})d\tilde{\xi}}_{=:p_\epsilon(\xi,d\tilde{\xi}),\text{ then } p_\epsilon(\xi,\Lambda)=\epsilon.}$$

Using the symmetry of $(1-\frac{1}{n}\Delta)^{-1}$ in $L^2(\Lambda)$ we obtain

$$\begin{aligned}\langle f, g-(1-\tfrac{1}{n}\Delta)^{-1}g\rangle &= \frac{1}{2}\int_\Lambda\int_\Lambda (f(\tilde{\xi})-f(\xi))(g(\tilde{\xi})-g(\xi))p_n(\xi,d\tilde{\xi})d\xi \\ &+ \int_\Lambda (1-(1-\tfrac{1}{n}\Delta)^{-1}1)fgd\xi.\end{aligned}$$

Idea of Proof

$$
\begin{aligned}
&\frac{1}{\epsilon}\langle \Phi(X_r), (1-\varepsilon\Delta)^{-1}X_r - X_r\rangle \\
&= -\frac{1}{2\varepsilon}\int_\Lambda\int_\Lambda [\Phi(X_r(\tilde{\xi})) - \Phi(X_r(\xi))][X_r(\tilde{\xi}) - X_r(\xi)]p_n(\xi, d\tilde{\xi})d\xi \\
&\quad -\frac{1}{\epsilon}\int_\Lambda (1-(1-\frac{1}{n}\Delta)^{-1}1)\Phi(X_r)X_r d\xi \\
&\leq -\frac{1}{2\varepsilon}\int_\Lambda\int_\Lambda (\zeta(X_r(\tilde{\xi})) - \zeta(X_r(\xi)))^2 p_n(\xi, d\tilde{\xi})d\xi \\
&\quad -\frac{1}{\epsilon}\int_\Lambda (1-(1-\frac{1}{n}\Delta)^{-1})\zeta(X_r)^2 d\xi \\
&= -\frac{1}{\epsilon}\langle \zeta(X_r), (1-(1-\frac{1}{n}\Delta)^{-1})\zeta(X_r)\rangle \\
&= -\mathcal{E}^{(\varepsilon)}(\zeta(X_r),\zeta(X_r)) \underbrace{\rightarrow}_{\text{Yosida approximation of the Dirichlet form}} -||\zeta(X_r)||^2_{H_0^1},
\end{aligned}
$$

where $(\mathcal{E}^{(\varepsilon)}, \mathcal{D}(\mathcal{E}^{(\varepsilon)}))$ is the Dirichlet form with generator $\frac{1}{\epsilon}(1-(1-\varepsilon\Delta)^{-1}) = \Delta(1-\varepsilon\Delta)^{-1}$.

Idea of Proof

Where we used the assumption:

- ζ-Weak monotonicity: For all $t, s \in \mathbb{R}$

$$(\Phi(t) - \Phi(s))(t - s) \geq (\zeta(t) - \zeta(s))^2.$$

- ζ-Coercivity: For p, a, c as in (A2) and for all $s \in \mathbb{R}$

$$\Phi(s)s \geq \zeta(s)^2 \geq a|s|^{p+1} - c.$$