Stochastic Flows induced by Stochastic Partial Differential Equations

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Introduction

- Principal aim: Understand the qualitative behavior of stochastic flows induced by SPDE

$$dX_t = A(t, X_t)dt + B_t(X_t)dN_t.$$

- A is a quasilinear, unbounded operator.
- In particular: Long-time behavior
  - Chaotic behavior,
  - Reduction of complexity.

(Random) Attractors

Reduction of complexity: Try to find a subset $\mathcal{A} \subseteq H$ such that

- $\mathcal{A}$ invariant.
- $\mathcal{A}$ captures the complete dynamics for large times, i.e. each trajectory can be approximated arbitrarily well by one lying in $\mathcal{A}$.
- $\mathcal{A}$ is minimal.

Then $\mathcal{A}$ is said to be a (random) attractor.

Stochastic Flows

- Markovian semigroup (dynamics of the distributions),
- Stochastic flow: $S(t,s;\omega) : H \to H$ such that
  - $S(s,s;\omega)x = x$,
  - Stochastic flow property: $S(t,s;\omega)x = S(t,r;\omega)S(r,s;\omega)x$,
- Cocycle property:

$$S(t,s;\omega)x = S(t-s,0;\theta_s\omega)x,$$

where $\theta_s : \Omega \to \Omega$ is a metric dynamical system, i.p. $(\theta_s)_* \mathbb{P} = \mathbb{P}$.

Overview

Contents:

- Strong solutions for SPDE of gradient type.
- Random attractors for stochastic porous media equations perturbed by space-time linear multiplicative noise.
- Random attractors for singular SPDE with general additive noise.
- Random attractors for degenerate SPDE perturbed by additive and real linear multiplicative noise.

Published papers:

- W-J. Beyn, G., P. Lescot, M. Röckner, *The Global Random Attractor for a Class of Stochastic Porous Media Equations*, CPDE, 2011.
- G., W. Liu, M. Röckner, *Random attractors for a class of stochastic partial differential equations driven by general additive noise*, JDE, 2011.

Part 1:
(Analytically) Strong Solutions for SPDE of Gradient Type

- In general: Solutions to SPDE are (spatially) less regular than to PDE, e.g. $dX_t = AX_t dt + f_t \mathrm{dt}$, then $X_t \in \mathcal{D}(A)$. While $dX_t = AX_t dt + dW_t$, then $X_t \in \mathcal{D}(A^{\frac{1}{2}})$.
- Common belief: Strong solutions to SPDE exist only in very special situations.

Theorem (Existence of strong solutions)

Let $X_0 \in L^2(\Omega; H)$. Then there exists a unique strong solution X to

$$dX_t = \underbrace{-\partial\varphi(X_t)}_{=A(X_t)} dt + B_t(X_t) dW_t$$

and $t^{\frac{1}{2}}\partial\varphi(X_t) \in L^2([0,T] \times \Omega; H)$. If $\mathbb{E}(\varphi(X_0)) < \infty$ then

$$\varphi(X) \in L^\infty([0,T]; L^1(\Omega))$$

- Unified framework: Stochastic porous medium equation, Stochastic p-Laplace equation, Stochastic reaction diffusion equation.

Nonlinear Galerkin approximation

- Idea of the proof:

$$\mathbb{E}\varphi(X_t) = \mathbb{E}\varphi(X_0) - \int_0^t \mathbb{E}\|\partial\varphi(X_r)\|_H^2 dr + \frac{1}{2}\int_0^t \mathbb{E}\, Tr(D^2\varphi(X_r)B(X_r)B^*(X_r))dr.$$

However: Lack of general Itô-formula in ∞-dimensions.

- Standard Galerkin approximation $P_n : H \to H_n$ is not compatible with the "geometry" induced by φ. E.g. $\varphi(P_n x) \not\leq C\varphi(x)$. Recall

$$\|P_n x - x\|_H = \inf_{y \in H_n} \|y - x\|_H.$$

- Main idea: Use Galerkin approximation weighted by intrinsic φ-distance. $\mathcal{P}_n : H \to H_n$ defined by

$$\varphi(\mathcal{P}_n x - x) = \inf_{y \in H_n} \varphi(y - x).$$

Then $\varphi(\mathcal{P}_n x) \leq 2\varphi(x)$.

Part 2:
Random Attractors for stochastic porous media equations perturbed by space-time linear multiplicative noise

Transformation into random PDE

We will consider:

$$dX_t = \Delta\left(|X_t|^m sgn(X_t)\right) dt + \sum_{k=1}^{N} f_k X_t \circ d\beta_t^k, \;\; m > 1,$$

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on a bounded domain $\mathcal{O} \subseteq \mathbb{R}^d$ with Dirichlet boundary conditions.

Generation of stochastic flows:

- Stochastic flow property: $S(t,s;\omega)x = S(t,r;\omega)S(r,s;\omega)x$,
- Does *not* follow from pathwise uniqueness,
- SPDE with additive noise: $dX_t = A(X_t)dt + dW_t$. Set $Y_t := X_t - W_t$, then
$$dY_t = A(Y_t + W_t)dt.$$
- Structural properties of A are preserved.

- Linear multiplicative space-time noise: $\mu := \sum_{k=1}^{N} f_k \beta_t^k$, $Y := e^{\mu} X$. Then
$$dY_t = e^{\mu_t} \Delta \left(e^{-m\mu_t} |Y_t|^m sgn(Y_t) \right).$$
- The transformed equation is not covered by any known existence and uniqueness results.
- In [BR10][1] for $Y_0 \in L^\infty(\mathcal{O})$ unique existence was shown in dimension $d \leq 3$.
- No continuity in the initial condition.
- Idea: Extend unique existence of solutions onto larger space ($\sim$ weaker norm) and prove continuity there.

Theorem

For $X_0 \in L^1(\mathcal{O})$ there is a unique **strong** *solution $X \in C([0,T]; L^1(\mathcal{O}))$ with*
$$\sup_{t\in[0,T]} \|X_t^{(1)} - X_t^{(2)}\|_{L^1(\mathcal{O})} \leq C \|X_0^{(1)} - X_0^{(2)}\|_{L^1(\mathcal{O})}.$$

[1] V. Barbu, M. Röckner, *On a random scaled Porous Medium Equation, JDE, 2011*

Bounds and regularity

For the existence of a random attractor one needs to show:

- Global asymptotic bounds (bounded absorption),
- Regularization (compact absorption).

Theorem

- *There is a piece-wise smooth function* $U : [0,T] \times \mathcal{O} \to \overline{\mathbb{R}}$ *such that*

$$X_t \leq U_t, \qquad \text{on } [0,T] \times \mathcal{O},$$

  independent of the initial condition X_0.
- X *is equicontinuous on every compact set* $K \subseteq (0,T] \times \mathcal{O}$.

Theorem

There is a random attractor $\mathcal{A}$ *(as a flow on* $L^1(\mathcal{O})$*).* $\mathcal{A}$ *is compact in* $L^\infty(\mathcal{O})$ *and is attracting in* L^∞*-norm.*

Part 3:
Random Attractors for Singular SPDE

Existence of (random) attractors

We consider

$$dX_t = A(t, X_t)dt + dN_t, \tag{3.1}$$

with

- Singular coercivity: ${}_{V^*}\langle A(v), v\rangle_V \leq -c\|v\|_V^\alpha + f_t$, $1 < \alpha < 2$.
- Noise: N_t càdlàg, strictly stationary increments, growth condition, e.g. Lévy process, fractional Brownian motion.
- Compactness: $A : V \to V^*$ with $V \subseteq H$ compact, e.g. Stochastic fast diffusion equations, singular stochastic p-Laplace equations (dimensional restrictions).

Theorem

- *There is a compact stochastic flow $S(t, s; \omega)x$ associated to (3.1) with a random attractor $\mathcal{A}$.*
- *If P_t is strongly mixing and $S(t, 0; \omega)x$ is monotone and contractive then $\mathcal{A}$ consists of a single stable equilibrium, i.e. $\mathcal{A}(\omega) = \{\eta(\omega)\}$.*

Part 4:
Random Attractors for Degenerate SPDE

Consider the SPDE:

$$dX_t = A(X_t)dt + dW_t + \mu X_t \circ d\beta_t. \tag{4.2}$$

- Simultaneous additive and real multiplicative noise
- Spatially rough noise.

Theorem

There is random attractor to (4.2) if

$$_{V^*}\langle A(t,v), v\rangle_V \leq -c\|v\|_V^\alpha + C\|v\|_H^2 + f_t, \quad \alpha \geq 2,$$

$V \subseteq H$ compact and there is a strongly monotone operator $M : V \to V^*$.

- Based on nonlinear Ornstein-Uhlenbeck processes.
- Improves semilinear results.
- Applies to: Stochastic generalized porous medium equation, stochastic generalized p-Laplace equations, stochastic generalized reaction diffusion equation.