Random attractors for stochastic porous media equations perturbed by space-time linear multiplicative noise

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Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded domain with smooth boundary $\partial \mathcal{O}$ in arbitrary dimension $d \in \mathbb{N}$, $T > 0$ and $\mathcal{O}_T := [0, T] \times \mathcal{O}$. We consider

$$dX_t = \Delta(|X_t|^m \text{sgn}(X_t))dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \quad \text{on } \mathcal{O}_T$$

(RPME)

$$X(0) = X_0, \quad \text{on } \mathcal{O}$$

$m > 1$, with Dirichlet boundary conditions, driving signals $z^{(k)} \in C([0, T]; \mathbb{R})$, $f_k \in C^\infty(\bar{\mathcal{O}})$. 

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Overview

1. Construction of a stochastic flow
   (a) Pathwise solution
   (b) Random case

2. Existence of Random Attractors
   (a) Bounded absorption
   (b) Asymptotic compactness & regularity of solutions
Part 1: Generation of Stochastic Flows

Problem: When do SPDE generate stochastic flows? Show

\[ X(t, s; \omega)x = X(t, r; \omega)X(r, s; \omega)x. \]

Technique: Transform the SPDE into an \( \omega \)-wise random PDE.

Obstacle: Depending on the structure of the noise the random PDE becomes difficult so solve. Works for:

1. additive noise \( B(X_t) = \text{const} \),
2. real linear multiplicative noise \( B(X_t) = cX_t \),
We consider linear multiplicative space-time noise

\[ dX_t = \Delta(|X_t|^m \text{sgn}(X_t))dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \text{ on } \mathcal{O}_T. \]

With \( \mu_t(\xi) = -\sum_{k=1}^{N} f_k(\xi)z_t^{(k)} \) the transformation \( Y = e^{\mu}X \) yields:

\[ dY_t = e^{\mu_t} \Delta(e^{-m\mu_t}|Y_t|^m \text{sgn}(Y_t)). \]  

(TPME)

Partial construction: [BR10]\(^1\).

\[ ^1\text{V. Barbu, M. Röckner, On a random scaled Porous Medium Equation, preprint, 2010} \]
Pathwise Solution

Theorem

*Essentially bounded distributional solutions to (TPME) are unique.*
Theorem

Let \( Y_0 \in L^1(\mathcal{O}) \), \( z \in C([0, T]; \mathbb{R}^N) \). There is a unique map \( Y \in C([0, T]; L^1(\mathcal{O})) \) satisfying

1. \( Y \in L^\infty([\tau, T] \times \mathcal{O}), \forall \tau > 0 \) and

\[
\frac{dY}{dt} = e^{\mu t} \Delta \Phi(e^{-\mu t} Y_t)
\]

for a.e. \( t \in [0, T] \) as an equation in \( H := (H^1_0(\mathcal{O}))^* \).

2. \( L^1 \)-contractivity:

\[
\sup_{t \in [0, T]} \| (Y_t^{(1)} - Y_t^{(2)})^+ \|_{L^1(\mathcal{O})} \leq C \| (Y_0^{(1)} - Y_0^{(2)})^+ \|_{L^1(\mathcal{O})}.
\]

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Random Case: Generation of RDS

Let \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) be a filtered probability space, \((z_t)_{t \in \mathbb{R}}\) be an \(\mathbb{R}^N\)-valued adapted stochastic process and \((\Omega, \mathcal{F}, \mathbb{P}), (\theta_t)_{t \in \mathbb{R}}\) be a metric dynamical system. Assume

(S1) (Strictly stationary increments) For all \(t, s \in \mathbb{R}, \omega \in \Omega:\)
\[ z_t(\omega) - z_s(\omega) = z_{t-s}(\theta_s \omega) - z_0(\theta_s \omega). \]

(S2) (Regularity) \(z_t\) has continuous paths.

We consider: \((\Phi(r) := |r|^m \text{sgn}(r))\)

\[
    dX_t = \Delta \Phi(X_t) dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \text{ on } \mathcal{O}_T \quad \text{(SPME)}
\]

\[ X(0) = X_0, \text{ on } \mathcal{O}. \]
Theorem

1. \( \varphi(t, \omega)x := X(t, 0; \omega)x = e^{-\mu t} Y(t, 0; \omega)x \) defines a continuous RDS on \( L^1(\mathcal{O}) \).

2. \( \varphi \) is quasi-weakly-continuous on \( L^p(\mathcal{O}) \), \( p \in [1, \infty) \) and quasi-weakly*-continuous on \( L^\infty(\mathcal{O}) \).

3. \( \varphi \) is order preserving (i.e. \( \varphi(t, \omega)x_1 \leq \varphi(t, \omega)x_2 \), a.e. in \( \mathcal{O} \) if \( x_1, x_2 \in L^1(\mathcal{O}) \) with \( x_1 \leq x_2 \) a.e. in \( \mathcal{O} \)).
Part 2: Random Attractors

Usual approach to prove existence of random attractors:

1. Prove $\omega$-wise attraction by a bounded set.
2. Based on this prove $\omega$-wise attraction by a compact set.

Recall

\[ dY_t = e^{\mu t} \Delta (\Phi(e^{-\mu t} Y_t)) \]

\[ = \Delta \left( e^{(1-m)\mu t} Y_t \right) - 2\nabla e^{\mu t} \cdot \nabla \Phi(e^{-\mu t} Y_t) - \Phi(e^{-\mu t} Y_t) \Delta e^{\mu t}. \]
Bounded absorption

Solution: Construct explicit supersolution with initial value $\infty$ and bounded for all $t > 0$. The construction combines an interval splitting technique from [BR10] and the known deterministic case:

$$K(t, \xi) = At^{-\frac{1}{m-1}}(R^2 - |\xi|^2)^{\frac{1}{m}}.$$ 

Theorem

There is a function $U : [0, T] \rightarrow \bar{\mathbb{R}}$ ($U(0) \equiv \infty$), piecewisely smooth on $(0, T]$ such that

$$|Y_t| \leq U_t, \text{ a.e. in } \mathcal{O},$$

for all $t \in [0, T]$, independent of the initial condition $Y_0 \in L^1(\mathcal{O})$.

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Asymptotic Compactness

We choose the approximations in a way that the results from [DB83] proving (local) uniform continuity of solutions are applicable.

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We say that a quantity depends only on the data if it is a function of \( d, m, T, \|X_0\|_{L^\infty(\mathcal{O})} \).

**Theorem**

Let \( z \in C([0, T]; \mathbb{R}^N) \), \( X_0 \in L^1(\mathcal{O}) \) and \( X \) be the corresponding solution. Then

1. \( X \) is continuous on every compact set \( K \subseteq (0, T] \times \mathcal{O} \), with modulus of continuity depending only on the data and \( \text{dist}(K, \partial\mathcal{O}_T) \).

2. Assume:

   \((\mathcal{O}1)\) \( \theta^* > 0, R_0 > 0 \) such that \( \forall x_0 \in \partial\mathcal{O} \) and \( \forall R \leq R_0 : |\mathcal{O} \cap B_R(x_0)| < (1 - \theta^*)|B_R(x_0)|. \)

   Then for every \( \tau > 0, X \) is continuous on \([\tau, T] \times \bar{\mathcal{O}}\) with modulus of continuity depending only on the data, \( \theta^* \) and \( \tau \).
In particular: The stochastic, variational solution $X$ to

$$dX_t = \Delta(|X_t|^m \text{sgn}(X_t))dt + \sum_{k=1}^{N} f_k X_k \circ d\beta^k_t$$

is $\mathbb{P}$-a.s. $L^\infty(\mathcal{O})$ bounded on each interval $[\tau, T]$, $\tau > 0$ and uniformly continuous on each compact set $K \subseteq (0, T] \times \mathcal{O}$. 
Theorem
The RDS \( \varphi \) has a random attractor \( A \) as an RDS on \( L^1(\mathcal{O}) \). \( A \) is compact and attracting in each \( L^p(\mathcal{O}) \), \( p \in [1, \infty) \).

\( A(\omega) \) is a bounded set in \( L^\infty(\mathcal{O}) \) and the functions in \( A(\omega) \) are equicontinuous on every compact set \( K \subseteq \mathcal{O} \).

If \((\mathcal{O}1)\) is satisfied, then \( A(\omega) \) is compact and attracting in \( L^\infty(\mathcal{O}) \).