Stochastic Partial Differential Equations and Random Dynamical Systems

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Gas Flow in a porous medium

Flow of an ideal gas in a homogeneous porous medium. ρ density, p pressure, v velocity

- Mass conservation:

$$\varepsilon\partial_t\rho + \nabla\cdot(\rho v) = 0,$$

- Darcy's law:

$$\mu v = -k\nabla p,$$

- State equation:

$$p = p_0\rho^\gamma,$$

where γ polytropic exponent, ε porosity of the medium, k permeability of the medium, p_0 reference pressure. We get:

$$\partial_t\rho = c\Delta(\rho^m).$$

Porous Medium Equation

General form:

$$\partial_t u = \Delta\Phi(u), \qquad (PME).$$

Applications:

- Nonlinear heat transfer (thermal conductivity depends on temperature).
- Phase transition (Stefan problem).
- Groundwater infiltration.
- Heat radiation in plasmas.
- Population dynamics (crowd avoiding).

PME and random environment

- Stochastic fluctuation of the permeability of the medium.
  Darcy's law: $\mu v = -k(\xi_t)\nabla p$.
- Model of the noise:
  - No further information on the correlations: ξ_{t_1}, ξ_{t_2} independent.
  - ξ_t identically distributed.
  - Central limit theorem: ξ_t normally distributed.
  - $\mathbb{E}[\xi_t \xi_s] = \delta(t-s)$.

  I.e. ξ is white noise.
- End up with

$$\partial_t \rho = c(\xi_t)\Delta(\rho^m), \qquad (RPME).$$

PME and random environment

- **Aim**: Derive stochastic properties of u from those of c.
- ξ-wise approach: Fix a realization of the noise ξ and solve (RPME) as a deterministic equation. Then derive information about distributions etc. from ξ-wise statements.

PME and SPDE

- **Aim**: Use probabilistic methods to study random PDE, e.g. consider corresponding Kolmogorov/Fokker-Planck equations.
- Much is known for SDE of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t.$$

- (So far) our equation has the form:

$$\frac{d}{dt}X_t = c(\xi_t)\Delta\Phi(X_t) = A(X_t, \xi_t).$$

PME and SPDE

- Expansion of first order in the noise:

$$\frac{d}{dt}X_t = A(X_t, 0) + D_\xi A(X_t, 0)\xi_t$$

- White noise is the "derivative" of Brownian motion:

$$\xi_t dt = \dot{W}(t)dt = dW_t.$$

- We get: (SPDE with multiplicative noise)

$$dX_t = A(X_t)dt + B(X_t)dW_t.$$

Note that this is an equation with infinitely many variables, i.e. on some infinite dimensional state space H. Usually H is some space of functions like $H = L^2(\mathcal{O})$.

Reduction to additive noise

- Fix the noise: $B \equiv B(Y_t)$ then study

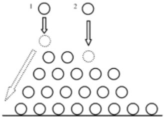
$$dZ_t = A(Z_t)dt + B(Y_t)dW_t,$$

  which is an SPDE with additive random noise.

- Denote by $\Gamma(Y) = Z$ the solution given the noise Y. Then recover the original solution X as a fixed point of Γ.

Self-organized criticality and SPDE

Bak-Tang-Wiesenfeld sandpile:



- Sand grains are dropped on a table at a random position.
- If the gradient of the density exceeds threshold an avalance occurs, i.e. rapid diffusion of mass.
- Model:

$$dX_t = div(\delta_{X_t=X_c} \nabla X_t)dt + dW_t = \Delta H(X_t \geq X_c)dt + dW_t.$$

- used to study chaotic phenomena (e.g. earthquakes, traffic jams).

Long-time behavior of SPDE

- Interested in invariant states.
- Reduction of complexity (due to egodicity or energy dissipation).

Dynamics of distributions

- Let $dX_t = b(X_t)dt + \sigma(X_t)dW_t$ on $H = \mathbb{R}^n$ "state space". Denote X_t^ν solution with initial distribution $X_0 \sim \nu$.
- Let $P_t\nu := (X_t^\nu)_*\mathbb{P}$ be the distribution of X_t^ν at time t, i.e.

$$P_t\nu(A) = \mathbb{P}[X_t^\nu \in A].$$

- Let $\mathcal{P} = Pr(H)$ be the set of all probability distributions over H. Then $P_t : \mathcal{P} \to \mathcal{P}$ is a (deterministic) dynamical system, i.e.

$$P_{t+s}\nu = P_t(P_s\nu),$$
$$P_t\nu \to \nu, \quad \text{for } t \to 0.$$

Dynamics of distributions

- P_t satisfies the Fokker-Planck equation:

$$\partial_t P_t \nu = \sum_{i,j=1}^{n} (\sigma(x)^t \sigma(x))_{i,j} \partial_i \partial_j P_t \nu + b(x) \cdot \nabla P_t \nu.$$

- Long-time behavior:

$$P_t \nu \to \mu, \text{ for } t \to \infty$$
$$P_t \nu \to \mathcal{A} \subseteq \mathcal{P}, \text{ where } \mathcal{A} \text{ is some "small", invariant set.}$$

Dynamics on the state space (RDS)

- $X_t : \Omega \to H = \mathbb{R}^n$. $t \mapsto X_t(\omega)$ is a random path in H.
- Problem: noise yields inhomogeneity in time

$$\text{“}dX_t = b(X_t)dt + \dot{W}(t)dt\text{“}.$$

Get an inhomogeneous semigroup.

Dynamics on the state space (RDS)

- Solution: Homogenization
  Let $\Omega = C_0(\mathbb{R};\mathbb{R}^n)$ and
  $$\begin{aligned}\theta_t : \Omega &\to \Omega\\ \omega &\mapsto \omega(t+\cdot)-\omega(t),\end{aligned}$$
  be the Wiener shift (ergodic dynamical system).
- Then
  $$\begin{aligned}\Pi_t : \Omega\times X &\to \Omega\times X\\ (\omega,x) &\mapsto (\theta_t\omega, X_t^x(\omega)).\end{aligned}$$
  is a dynamical system.
- The flow property for X becomes
  $$X_{t+s}^x(\omega) = X_t^{X_s^x(\omega)}(\theta_t\omega)$$
  which is called cocycle property and X a random dynamical system (RDS).

Attraction

Reduction of complexity of the dynamics due to long-time evolution. For P_t on $\mathcal{P}$ (Π_t on $\Omega \times H$) we try to find a subset $A \subseteq \mathcal{P}$ ($A \subseteq \Omega \times H$ resp.) such that

- A invariant.
- A captures the complete dynamics for large times, i.e. each trajectory can be approximated arbitrarily well by one lying in A.
- A is minimal.

Then A is said to be an attractor for P_t (Π_t resp.).

Comparison of approaches

Assertions for RDS (Π_t) are usually stronger. E.g. attractor for RDS (Π_t) implies attractor for distributional dynamics (P_t).

Main obstacles

- Hard to show that a solution X_t to an SPDE gives an RDS i.e. that
$$X^x_{t+s}(\omega) = X_t^{X^x_s(\omega)}(\theta_t\omega),$$
because for SPDE X_t is usually only well-defined and unique $\mathbb{P}$-a.s.
- Only existing technique: Transform SPDE into RPDE
$$dX_t = A(X_t)dt + BdW_t, \qquad (SPDE).$$
Let $Y_t = X_t - BW_t$, then
$$dY_t = A(Y_t + BW_t)dt, \qquad (RPDE).$$

Main obstacles

- Prove dissipation: X_t^x lies in a bounded set of the state space independent of the initial condition x after large enough times t.
- Prove regularization: e.g. for $x \in L^2(\mathcal{O})$ the solution has more differentiability $X_t^x \in H_0^1(\mathcal{O})$.