Random attractors for a class of stochastic partial differential equations driven by general additive noise

Benjamin Gess
University of Bielefeld

joint work with:
Wei Liu
Michael Röckner

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Outline

Motivation

Motivation

Why study long-time behaviour?

At least two crucial aims:

- Understand chaotic behaviour, turbulence
- Reduction of complexity (ergodicity or dissipation of energy)

Reduction of complexity

'Fully developed' dynamics (i.e. that have been run a long time) have reduced complexity. There exists a 'small' set $A \subseteq H$ (H the state space) such that

- A is invariant under the flow
- Each trajectory can be approximated arbitrarily well by one lying in A (for large times)
- A is minimal.

Sets with these properties are called attractors.

Basics on RDS

Basics on Random Dynamical Systems (RDS)

Basics on RDS

Definition (1.1)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta_t : \Omega \to \Omega$, $t \in \mathbb{R}$, be a family of maps. Then $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ is said to be a **metric dynamical system** if

- $\theta : \mathbb{R} \times \Omega \to \Omega$ is measurable
- $\theta_0 = id$, $\theta_{t+s} = \theta_t \circ \theta_s$
- $(\theta_t)_* \mathbb{P} = \mathbb{P}$

e.g. $\Omega = C_0(\mathbb{R}; \mathbb{R})$, $\mathbb{P} =$ Wiener measure, $\theta_t(\omega) = \omega(t + \cdot) - \omega(t)$ ("Wiener shift").

Basics on RDS

Definition (1.2)

Let (H, d) be a complete and separable metric space, $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ a metric dynamical system and $\varphi : \mathbb{R}_+ \times \Omega \times H \to H$ measurable with

- $\varphi(0, \omega) = \mathrm{id}$
- $\varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega)$ (cocycle property)
- $\varphi(t, \omega) : H \to H$ continuous.

Then φ is called a **random dynamical system** (RDS)

(e.g. $\varphi(t, \omega)x = X(t, 0; \omega)x$ the solution to a stochastic equation).

Definition (1.3)

- $K : \Omega \to 2^H$ is called **measurable** if $K(\omega)$ is closed and $\omega \mapsto d(x, K(\omega))$ is measurable for all $x \in H$, where d is the Hausdorff semidistance. K is also called a **random set**.

Basics on RDS

Definition (1.3)

- Let A, B be random sets. A is said to **absorb** B if for $\mathbb{P}$-a.e. $\omega \in \Omega$ there exists an **absorption time** $t_B(\omega)$ such that for all $t \geq t_B(\omega)$

$$\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subseteq A(\omega).$$

- A is said to **attract** B if

$$d(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)) \xrightarrow[t \to \infty]{} 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Basics on RDS

Definition (1.4)

A **random attractor** (RA) for an RDS φ is a random set A satisfying $\mathbb{P}$-a.s.

- (invariance): $\varphi(t,\omega)A(\omega) = A(\theta_t\omega)$ for all $t > 0$.
- (attraction): A attracts all deterministic bounded sets $B \subseteq H$, i.e.

$$d(\varphi(t,\theta_{-t}\omega)B, A(\omega)) \xrightarrow[t\to\infty]{} 0.$$

- (compactness): $A(\omega)$ is compact.

Some known results

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Basics on RDS

Some known results for RA: (not exhaustive)

- Only semilinear equations. $dX_t = (AX_t + F(X_t))dt + B(X_t)dW_t$. But quasilinear equations (e.g. PME) produce interesting dynamical behaviour:
  - Polynomial decay of solutions,
    e.g. for PME $|u(t)| \le (|u(0)|^{-(p-1)} + Ct)^{-\frac{1}{p-1}}$.
  - Finite time extinction, e.g. FDE: $|u(t)| \le (|u(0)|^{1-p} - Ct)^{\frac{1}{1-p}}$.
  - Free boundaries ($\mathrm{supp} u(t) \subseteq \Lambda$).
- Only result for non-semilinear equations: [Beyn, G., Lescot, Röckner, CPDE, 2011].

Definition (The Stochastic Porous Medium Equation (SPME))

On a bounded, open set $\Lambda \subseteq \mathbb{R}^d$:

$$dX_t = \Delta(|X_t|^{p-1}X_t)dt + dN_t,$$

with Dirichlet boundary conditions.

Basics on RDS

Some known results for RA: (not exhaustive)

- Mostly Brownian noise. RDS approach especially interesting for non-Markovian processes (no associated semigroup) like fractional Brownian Motion (fBM).
- For fBM:
  - [Garrido-Atienza, Kloeden, Neuenkirch, AMO, 2009] (SDE, strong monotone drift, singleton attractors)
  - [Maslowski, Schmalfuss, SAA, 2004] (Semilinear SPDE, Hurst parameter $> \frac{1}{2}$, strong monotone drift, singleton attractor).

RDS given by SPDE driven by additive noise

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RDS given by SPDE driven by additive noise

- Let
$$V \subseteq H \equiv H^* \subseteq V^*$$
be a Gelfand triple, $A : V \to V^*$ be measurable.
- $(N_t)_{t\in\mathbb{R}}$ a V-valued adapted stochastic process.
- For $[s,t] \subseteq \mathbb{R}$ we consider the stochastic evolution equation

$$\begin{aligned} dX_r &= A(X_r)dr + dN_r, \; r \in [s,t], \qquad \text{(SPDE)} \\ X_s &= x \in H. \end{aligned}$$

e.g. for SPME $V = L^{p+1}(\Lambda)$, $H = (H_0^1(\Lambda))^*$, $A(v)(w) = -\int_\Lambda vwd\xi$.

RDS given by SPDE driven by additive noise

Definition

An H-valued, $(\mathcal{F}_t)$-adapted process $\{X_r\}_{r\in[s,t]}$ is called a solution of $(SPDE)$ if $X_\cdot(\omega) \in L^\alpha([s,t];V) \cap L^2([s,t];H)$ and

$$X_r(\omega) = x + \int_s^r A(X_\tau(\omega))d\tau + N_r(\omega) - N_s(\omega)$$

holds for all $r \in [s,t]$ and all $\omega \in \Omega$.

RDS given by SPDE driven by additive noise

By "shifting the noise" SPDE reduces to random PDE with time and ω-dependent coefficients, i.e. change of variables

$$X_t \rightsquigarrow X_t - N_t =: Z_t$$

yields (in integral form) for $\mathbb{P}$-a.e. $\omega \in \Omega$

$$Z_t(\omega) = x - N_s(\omega) + \int_s^t A_\omega(r, Z_r(\omega))dr, \quad t \geq s, \quad \text{(RPDE)}$$

where $x \in H$ and

$$A_\omega(r, \cdot) := A(\cdot + N_r(\omega)).$$

Then

$$\varphi(t,\omega)x := X(t,0;\omega)x := Z(t,0;\omega)x + N_t(\omega), \quad t \geq 0,$$

defines an RDS.

Assumptions on the drift

Suppose that there exists $\alpha > 1$ and constants $\delta > 0$, $K, C \in \mathbb{R}$ such that the following conditions hold for all $v, v_1, v_2 \in V$:

$(H1)$ (Hemicontinuity) The map $s \mapsto {}_{V^*}\langle A(v_1 + sv_2), v\rangle_V$ is continuous on $\mathbb{R}$.

$(H2)$ (Monotonicity)

$$2\,{}_{V^*}\langle A(v_1) - A(v_2), v_1 - v_2\rangle_V \leq C\|v_1 - v_2\|_H^2.$$

$(H3)$ (Coercivity)

$$2\,{}_{V^*}\langle A(v), v\rangle_V + \delta\|v\|_V^\alpha \leq C + K\|v\|_H^2.$$

$(H4)$ (Growth)

$$\|A(v)\|_{V^*} \leq C(1 + \|v\|_V^{\alpha-1}).$$

Assumptions on the noise

Let $((\Omega, \mathcal{F}, \mathbb{P}), (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system.

$(S1)$ (Strictly stationary increments) For all $t, s \in \mathbb{R}$, $\omega \in \Omega$:

$$N_t(\omega) - N_s(\omega) = N_{t-s}(\theta_s\omega) - N_0(\theta_s\omega).$$

$(S2)$ (Regularity) For each $\omega \in \Omega$,

$$N_\cdot(\omega) \in L^\alpha_{loc}(\mathbb{R}; V) \cap L^2_{loc}(\mathbb{R}; H)$$

(with the same $\alpha > 1$ as in $(H3)$).

$(S3)$ (Joint measurability) $N : \mathbb{R} \times \Omega \to V$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}/\mathcal{B}(V)$ measurable.

RDS given by SPDE driven by additive noise

Theorem

Under the assumptions $(H1)$-$(H4)$ and $(S1)$-$(S3)$, then φ is a continuous random dynamical system.

Existence of Random Attractors

Existence of Random Attractors

Proposition (1.6 [Crauel/Flandoli: PTRF 1994])

Let φ be an RDS and assume the existence of a compact random set K absorbing every deterministic bounded set $B \subseteq H$. Then there exists a random attractor A.

Existence of Random Attractors

How to find a compact, globally absorbing random set K:

- Let $S \subseteq H$ be a compactly embedded subspace, such that $V \subseteq S \subseteq H$. K will be chosen as

$$K(\omega) := \overline{B_S(0, r(\omega))}^H .$$

- Note that

$$\varphi(t, \theta_{-t}\omega) = X(t, 0; \theta_{-t}\omega) = X(0, -t; \omega).$$

  K absorbing means

$$||X(0, -t; \omega)x||_S \leq r(\omega),$$

  for all t large enough.

- Need pathwise bounds on $X(0, -t; \omega)x$ in the S-norm.

e.g. for SPME $V = L^{p+1}(\Lambda) \subseteq S = L^2(\Lambda) \subseteq H = (H_0^1(\Lambda))^*$.

Existence of Random Attractors

To show such a bound we proceed in two steps:

- Bounded absorption: Bound for the H-norm

$$\frac{d}{dt}\|Z_t\|_H^2 + \frac{\delta_0}{2}\|Z_t\|_V^\alpha \leq -\lambda\|Z_t\|_H^2 + f_t + C, \quad (\star)$$

  for some $\alpha > 0$ and some function f_t of subexponential growth.
- Compact absorption: Bound for stronger norm $(||\cdot||_S)$

$$\frac{d}{dt}\|Z_t\|_S^2 \leq C\|Z_t\|_V^\alpha + g_t, \quad (\star\star)$$

  for some function $g_t \in L^1_{loc}(\mathbb{R})$.

Existence of Random Attractors - Compact absorption

How do prove $(\star\star)$:

- Want to apply Itô's formula to $||X||_S^2$.
- main idea: Approximate $||\cdot||_S$-norm by norms $\|\cdot\|_n$ equivalent to $\|\cdot\|_H$ such that for all $x \in S$ we have $\|x\|_n \uparrow \|x\|_S$. And suppose that $\|\cdot\|_n$ are given via $\langle x, y\rangle_n := \langle x, T_n y\rangle_H$.
- E.g. $T_n = -\Delta(1 - \frac{\Delta}{n})^{-1}$ the Yosida approximation of the Laplace operator on L^2. Then $\langle x, y\rangle_n$ is the Yosida approximation of the Dirichlet form corresponding to Δ on L^2 and $||x||_{L^2} \sim ||x||_n \uparrow ||x||_{H_0^1}$.

Existence of Random Attractors - Compact absorption

- Approximate inequality:

$$\begin{aligned}\frac{d}{dt}\|Z_t\|_n^2 &= 2\,{}_{V^*}\langle A(Z_t+N_t), T_nZ_t\rangle_V \\ &\leq C(\|Z_t+N_t\|_n^2+1) - 2\,{}_{V^*}\langle A(Z_t+N_t), T_nN_t\rangle_V \\ &\leq C\left(\|Z_t\|_n^2+\|Z_t\|_V^\alpha\right) + C\left(1+\|N_t\|_n^2+\|N_t\|_V^\alpha+\|T_nN_t\|_V^\alpha\right) \\ &\leq C\left(\|Z_t\|_n^2+\|Z_t\|_V^\alpha\right) + g_t^{(n)},\end{aligned}$$

- Applying Gronwall, then taking $n \to \infty$ yields the needed bound.

Existence of Random Attractors

$(H5)$ Let T_n be positive definite self-adjoint on H and let $\langle x, y\rangle_n := \langle x, T_n y\rangle_H$, H. Suppose $\|\cdot\|_n \sim \|\cdot\|_H$ and

$$\|x\|_n \uparrow \|x\|_S \text{ as } n \to \infty, \ \forall x \in S.$$

Assume that $T_n : V \to V$ continuous and that $\exists C > 0$ such that

$$2_{V^*}\langle A(v), T_n v\rangle_V \leq C(\|v\|_n^2 + 1), \ v \in V,$$

and

$$\sup_{n\in\mathbb{N}} \int_{-1}^{0} \|T_n N_t\|_V^\alpha dt \leq C.$$

$(S4)$ (Subexponential growth) For $\mathbb{P}$-a.a. $\omega \in \Omega$ and $|t| \to \infty$, $N_t(\omega)$ is of subexponential growth, i.e. $\|N_t(\omega)\|_V = o(e^{\lambda|t|})$ for every $\lambda > 0$.

Existence of Random Attractors

Theorem

Suppose $(H1)$-$(H5)$ *hold for* $\alpha = 2, K = 0$ *or for* $\alpha > 2$*, and that* $(S1)$-$(S4)$ *are satisfied. Then the RDS* φ *has a compact random attractor.*

Monotonicity and Singleton Attractors

Monotonicity and Singleton Attractors

Monotonicity and Singleton Attractors

- Monotonicity of the drift drives trajectories together:

$$X(t,0;\omega)x - X(t,0;\omega)y = x - y + \int_0^t A(X(r,0;\omega)x) - A(X(r,0;\omega)y)dr.$$

- If the monotonicity is strong enough, the whole state space is contracted to a single point (possibly time-dependent, resp. random).
- Itô's formula:

$$\begin{aligned} &||X(t)x - X(t)y||_H^2 \\ &= ||x-y||_H^2 + \int_0^t < A(X(r)x) - A(X(r)y), X(r)x - X(r)y > dr \\ &\leq ||x-y||_H^2 - \lambda \int_0^t ||X(r)x - X(r)y||_H^\beta dr. \end{aligned}$$

Monotonicity and Singleton Attractors

Theorem

Suppose that $(H1)$,$(H2')$,$(H3)$,$(H4)$ *and* $(S1)$-$(S3)$ *hold. If* $\beta = 2$ *also suppose* $(S4)$ *holds. Then the RDS* φ *has a compact random attractor* $\mathcal{A}(\omega)$ *consisting of a single point:*

$$\mathcal{A}(\omega) = \{\eta_0(\omega)\}.$$

In particular, there is a unique invariant random measure $\mu_. \in \mathcal{P}_\Omega(H)$ *which is given by*

$$\mu_\omega = \delta_{\eta_0(\omega)}, \quad \mathbb{P}\text{-a.s.} \ .$$

Monotonicity and Singleton Attractors

For the speed of attraction we have:

(i) if $\beta > 2$, then the speed of convergence is polynomial, more precisely,

$$\|X(t,s;\omega)x - \eta_0(\theta_t\omega)\|_H^2 \leq \left\{\frac{\lambda}{2}(\beta-2)(t-s)\right\}^{-\frac{2}{\beta-2}}, \ \forall x \in H.$$

(ii) if $\beta = 2$, then the speed of convergence is exponential. More precisely, for every $\eta \in (0,\lambda)$ there is a random variable K_η such that

$$\|X(t,s;\omega)x - \eta_0(\theta_t\omega)\|_H^2 \leq 2\left(K_\eta(\omega) + \|x\|_H^2\right) e^{(\lambda-\eta)s} e^{-\lambda t}, \ \forall x \in H.$$

Applications

Applications

1. Admissible Noise

Admissible Noise

Lemma

Let $(N_t)_{t\in\mathbb{R}}$ be a V-valued process with stationary increments and a.s. càdlàg paths. Then there is a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ and a version $\tilde{N}_t$ on $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ such that $\tilde{N}_t$ satisfies $(S1)$-$(S3)$.

Proof:
Take $\Omega = D(\mathbb{R}; V)$ to be the set of all càdlàg functions endowed with the Skorohod topology, $\mathcal{F} = \mathcal{B}(\Omega)$, $\theta_t(\omega) = \omega(t + \cdot) - \omega(t)$ and $\mathbb{P} = \mathcal{L}(N)$ to be the law of N_t.

Admissible Noise

Lemma

Let V be a separable Banach space and N_t be a V-valued Lévy process with Lévy characteristics (m, R, ν). Assume $\int_V \left(\|x\|_V \vee \|x\|_V^2\right) d\nu(x) < \infty$, then we have $\mathbb{P}$-a.s.

$$\frac{N_t}{|t|} \to \pm \mathbb{E} N_1 \ (t \to \pm\infty).$$

Admissible Noise

Lemma

Let $(N_t)_{t\in\mathbb{R}}$ be a strictly stationary V-valued process on a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$. Assume $\exists\gamma > 1$, $\alpha > 0$ and $C \in \mathbb{R}$ such that

$$\mathbb{E}\|N_t - N_s\|_V^\gamma \le C|t-s|^{1+\alpha}, \ \forall t, s \in \mathbb{R}.$$

Then there exists a θ_t-invariant set $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ and for any $\epsilon > 0$, $\omega \in \Omega_0$, there exists a constant $C_1 = C_1(\epsilon, \omega)$ such that

$$\|N_t(\omega)\|_V \le \epsilon|t|^2 + C_1, \ \forall t \in \mathbb{R}.$$

In particular, N_t satisfies $(S4)$.

For example: $N_t = B_t^H$ fractional Brownian Motion with Hurst parameter $H \in (0,1)$, trace class covariance with sufficiently fast decaying coefficients.

Admissible Drifts

Applications

2. Admissible Drifts

Admissible Drifts

- Stochastic reaction-diffusion equation:

$$dX_t = (\Delta X_t - |X_t|^{p-2} X_t + \eta X_t) dt + dN_t,$$

- Stochastic porous media equation:

$$dX_t = \left(\Delta(|X_t|^{r-1} X_t) + \eta X_t\right) dt + dN_t,$$

- Stochastic p-Laplace equation:

$$dX_t = \left[\mathbf{div}(|\nabla X_t|^{p-2} \nabla X_t) - \eta_1 |X_t|^{\tilde{p}-2} X_t + \eta_2 X_t\right] dt + dN_t.$$