

# Finite speed of propagation for stochastic porous media equations.

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# Outline

- 1 The deterministic case
- 2 The stochastic case

# The deterministic case

**The deterministic case**

# The Deterministic PME

- Recall: The deterministic porous medium equation

$$\frac{d}{dt}u = \Delta(u^m), \quad (1)$$

for non-negative initial conditions  $u_0 \geq 0$  [Vázquez, 2006].

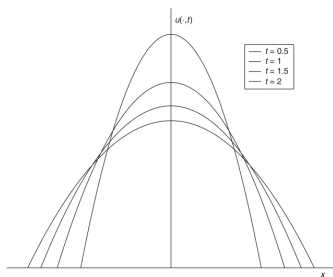
- In the superlinear case  $m > 1$ , (1) has degenerate diffusivity:

$$\frac{d}{dt}u = mu^{m-1}\Delta u + m(m-1)u^{m-2}|\nabla u|^2.$$

The diffusivity coefficient vanishes for  $u \rightarrow 0$ .

# Finite speed of propagation

- Limited regularity:  $\nabla u$  discontinuous. E.g. Barenblatt solutions:



- Regularity is limited precisely at the free boundary.
- Comparison to Barenblatt solutions yields finite speed of propagation.

# Optimal estimates

- Finite speed of propagation is a local property.
- *Finite speed of hole-filling:*  
Let  $u_0$  vanish in  $B_R(x_0)$ . Then  $u$  vanishes in  $B_{R(t)}(x_0)$  with

$$R(t) \geq R - \left( \frac{\|u\|_\infty^{m-1}}{C_{det}} t \right)^{1/2}.$$

This is optimal.

- *Finite speed of propagation:*  
Let  $S(t) := \{x \in \mathbb{R}^d \mid u(t, x) > 0\}$  be the positivity set. Then

$$S(t+h) \subseteq B_{ch^{1/2}}(S(t)).$$

# The stochastic case

**The stochastic case**

# Stochastic porous medium equation

- Stochastic porous medium equation

$$dX_t = \Delta X_t^m dt + \sum_{k=1}^N f_k X_t \circ d\beta_t^{(k)}. \quad (\text{SPME})$$

on bounded domains, homogeneous Dirichlet boundary conditions.

- $\beta^k$  Brownian motion,  $f_k \in C^\infty(\mathcal{O})$ .



# Known results

- *Finite speed of hole-filling* [Barbu, Röckner, EJP 2012]:  
Let  $X_0$  vanish in  $B_R(x_0)$ . Then  $X$  vanishes in  $B_{R(t,\omega)}(x_0)$  for some function  $R(\cdot, \omega) : [0, T] \rightarrow (0, R)$ .
- No uniform control on  $R(t, \omega)$  in  $x_0 \rightarrow$  cannot deduce finite speed of propagation
- No information about optimality of the bounds
- Two aims:
  - show finite speed of propagation
  - deduce (locally) optimal bounds

# Transformation

- Recall:

$$dX_t = \Delta(X_t^m) dt + \sum_{k=1}^N f_k X_t \circ d\beta_t^{(k)}.$$

- Set  $Y_t := e^{\mu t} X_t$ , where  $\mu_t = -\sum_{k=1}^N f_k \beta_t^{(k)}$ . Then

$$\partial_t Y(t, x) = e^{\mu(t, x)} \Delta \left( e^{-\mu(t, x)} Y(t, x) \right)^m.$$

- Existence and uniqueness in [Barbu, Röckner, JDE, 2011], [Gess, 2011, arXiv:1108.2413].

## Non-spatially distributed noise

- Assume  $f_k$  constant. Then

$$\partial_t Y_t = e^{(1-m)\mu t} \Delta Y_t^m.$$

- Let  $F' := e^{(1-m)\mu t}$ ,  $g = F^{-1}$ . Then  $u_t := Y_{g(t)}$  solves

$$\frac{d}{dt} u = \Delta(u^m)$$

- Finite speed of propagation follows from the deterministic case and the estimates are **optimal**.
- Finite speed of hole-filling:*

Let  $X_0$  vanish in  $B_R(x_0)$ . Then  $X_t$  vanishes in  $B_{R_{stoch}(t,\omega)}(x_0)$  with

$$\begin{aligned} R_{stoch}(t) &= R - \left( \frac{H^{m-1}}{C_{det}} F(t) \right)^{\frac{1}{2}} \\ &= R - \left( \frac{H^{m-1}}{C_{det}} \int_0^t e^{-(m-1)\mu_r(\omega)} dr \right)^{\frac{1}{2}}. \end{aligned}$$

# Spatially distributed noise

- Recall:

$$\partial_t Y(t, x) = e^{\mu(t, x)} \Delta \left( e^{-\mu(t, x)} Y(t, x) \right)^m.$$

- freeze coefficients in space:

$$\partial_t Y(t, x) \approx e^{\mu(t, x_0)} \Delta \left( e^{-\mu(t, x_0)} Y(t, x) \right)^m,$$

on small balls  $B_r(x_0)$ .

- freeze coefficients in time:

$$\partial_t Y(t, x) \approx \Delta Y(t, x)^m,$$

for small times  $t \approx 0$ .

# Hole-filling

- *Finite speed of hole-filling:*

Let  $X_0$  vanish in  $B_R(x_0)$ . Then  $X_t$  vanishes in  $B_{R_{stoch}(t,\omega)}(x_0)$  with

$$\begin{aligned} R_{stoch}(t, \omega) &= R - \left( \frac{H^{m-1}}{C_{det}} F(t, \omega) \right)^{\frac{1}{2}} C_R^{-\frac{1}{2}}(\omega) \\ &= R - \left( \frac{H^{m-1}}{C_{det}} \int_0^t e^{-(m-1)\mu_r(x_0, \omega)} dr \right)^{\frac{1}{2}} C_R^{-\frac{1}{2}}(\omega), \end{aligned}$$

with  $\lim_{R \downarrow 0} C_R = 1$ .

- For  $R \approx 0$  we recover the optimal rate from the spatially homogeneous case with  $\mu_r \equiv \mu_r(\xi_0)$ .

# Hole-filling

- *Finite speed of hole-filling:*

Let  $X_0$  vanish in  $B_R(x_0)$ . Then  $X_t$  vanishes in  $B_{R_{stoch}(t,\omega)}(x_0)$  with

$$R_{stoch}(t, \omega) = R - \left( \frac{H^{m-1}}{C_{det}} t \right)^{\frac{1}{2}} \sqrt{C_t(\omega)}.$$

with  $\lim_{t \downarrow 0} C_t = 1$ .

- For  $t \approx 0$  we recover the optimal rate from the deterministic case.

# Finite speed of propagation

- *Finite speed of propagation:*

Let  $X$  be an essentially bounded, non-negative solution to (SPME). Then,

$$\text{supp}(X_t) \subseteq B_{\sqrt{t} \left( \frac{H^{m-1}}{C_{\det}} \right)^{\frac{1}{2}} \sqrt{C_t(\omega)}}(\text{supp}(X_0)), \quad \forall t \in [0, T],$$

with  $C_t \rightarrow 1$  for  $t \rightarrow 0$ .