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Qualitative Behaviour of Stochastic Evolution Equations

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1. Introduction

The qualitative study of evolutionary systems is a fundamental tool in the analysis of mathematical models. Many dynamical systems (DS) exhibit a far too complex behaviour to allow a detailed study of single trajectories. This is why instead of concentrating on the evolution of a single initial state one rather considers geometric properties of the phase portrait, such as invariant manifolds and attractors.

In many situations the dynamics of an evolutionary system become stationary or at least simpler after some time. This reduction of complexity can for example be explained by the dissipation of energy and mixing properties of the system (ergodicity). For systems showing chaotic behaviour it is therefore promising to restrict the attention to the long time behaviour.

The attractor of a DS is a set A in the state space, which on the one hand captures the complete long-time behaviour of the system, while on the other hand being as small as possible. Thus, concerning the long-time dynamics one can essentially restrict the DS to this set A on which the behaviour of the solutions often is much easier.

One of the most prominent examples of this approach are the 2D-Navier Stokes equations. While initially the flow shows very complex behaviour eventually it 'settles down'. This behaviour is crucial for windtunnel experiments since only by the reduction of complexity measurements become feasible. In mathematical terms this effect has been shown by the existence of a finite-dimensional (Hausdorff-dimension) attractor, which in fact can be embedded into a finite dimensional space. This can be seen as a proof that for fully developed dynamics finitely many measurements may determine the whole state.

When applying the ideas of dynamical systems and their long-time behaviour to systems generated by stochastic differential equations the term of a dynamical system has to be generalized. Since the random perturbation may depend on time, the resulting system is intrinsically nonautonomous. The theory of random dynamical systems makes use of the fact that this inhomogeneity is not just an arbitrary one but can be expressed in terms of the randomness. This leads to the so-called cocycle property which can be seen as a generalization of the flow property of deterministic dynamical systems.

2. Framework and known results

Since we are interested in the dynamical behaviour of solutions to highly nonlinear equations, our study is based on the variational approach to stochastic partial differential equations (SPDE). Hence we sudy evolution equations of the form

$$dX_t = A(X_t)dt + B(X_t)dW_t,$$

where $A : V \to V^*$ is a monotone operator. The theory of existence and uniqueness of solutions to this kind of equations, under certain coercivity and boundedness conditions is well known (cf. [5]).

The dynamical counterpart of stochastic (partial) differential equations is given by the theory of random dynamical systems (RDS) (cf. [1]). The fundamental property of an RDS $\varphi : \mathbb{R}_+ \times \Omega \times H \to H$ is a stochastic version of the flow propery, named (perfect) cocycle property:

$$\varphi(t+s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega),$$

for all $t, s \geq 0, \omega \in \Omega$. The relation between the solution X_t to the stochastic equation and the RDS φ is then given by:

$$\varphi(t, \omega)x := X(t, 0, x)(\omega),$$

where $X(t, s, x)$ denotes the value of the solution at time t when started in x at time $s \leq t$.

While for ordinary stochastic differential equations this property has been proved for a large class of equations, for SPDEs it could so far only be obtained in special cases.

One way to capture the long-time behaviour of a RDS is to consider its random attractor (cf. [3]). A random attractor $A : \Omega \to 2^H$ is a compact random set satisfying

- (invariance): $\varphi(t, \omega)A(\omega) = A(\theta_t\omega)$
- (pullback attraction): $d(\varphi(t, \theta_t\omega)B, A(\omega)) \to 0$, for all $B \subseteq H$ bounded.

Here $d(A, B) := \sup_{a \in A} \inf_{b \in B} ||a - b||$ is the Hausdorff semi-metric of sets.

The existence of the global random attractor for several semilinear SPDEs is known. In the case that the random attractor consists of a single (random) point, the convergence of the stationary solution of the discretized system (implicit Euler, Galerkin) to this attractor has been shown for semilinear, monotone equations.

3. New results

- The porous medium equation (PME)

  $$dX_t = \Delta\Big(|X_t|^p sgn(X_t)\Big)dt, \quad p > 1$$

  is a heavily used model of the evolution of densities in porous media. One important example is the modelling of oil reservoirs. Since the stochastic porous medium equation (SPME) is a quasilinear equation with superlinear coercivity, the long-time behaviour is intriguing and deviates significantly from semilinear SPDEs.

  In [2] a new strong inequality for the L^2 norm of solutions to the generalized SPME has been proved and used to deduce the existence of a random attractor. The generalized SPME is of the form

  $$dX_t = \Delta\Phi(X_t)dt + dW_t,$$

  where $\Phi : \mathbb{R} \to \mathbb{R}$ is a monotone function satisfying some coercivity and boundedness properties. In case of a strongly monotone nonlinearity Φ the attractor has been shown to be a singleton.
- In the literature the existence of random attractors has been shown for specific examples of semilinear equations, driven by specific kinds of noise (either Brownian or fractional Brownian noise). In [4] an abstract result proving the existence and structure of random attractors is obtained and applied to several quasilinear equations (SPME, p-Laplace, Reaction Diffusion). The analysed equations are of the form

  $$dX_t = A(X_t)dt + dN_t,$$

  where $A : V \to V^*$ is a monotone operator and N_t is some stochastic perturbation. General conditions for the admissible additive random perturbation, applicable to Levy processes and infinite-dimensional fractional Browian Motion are given. In case of a strongly monotone drift optimal bounds for the speed of convergence towards the singleton attractor are obtained. This is of particular importance for numerical approximation of the attractor.

4. Prospects

- Although being of the same structure as the PME the fast diffusion equation (FDE)

  $$dX_t = \Delta\Big(|X_t|^p sgn(X_t)\Big)dt, \quad p < 1$$

  exhibits quite a different long-time behaviour. The solutions reach 0 after some finite time. Due to the low coercivity (order less than 2) the dynamics of the stochastically perturbed version are expected to be interesting. The existence and structure of the corresponding random attractor is under investigation.
- Several examples show, that while additive noise can regularize the dynamics (e.g. yield a single-point attractor), multiplicative noise preserves the dynamic complexity of the corresponding deterministic equation. We are therefore studying the extension of the results obtained in [2, 4] to the case of multiplicative noise.
- The convergence of numerical methods for semilinear equations with a unique stationary solution is well known. The generalization to quasilinear equations (like SPME, FDE) is under investigation.

References

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