

# Stochastic PDE, non-equilibrium fluctuations and large deviations

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joint work with: Ben Fehrman, Nicolas Dirr.  
[Fehrman, G.; arxiv, 2020], [Dirr, Fehrman, G.; arxiv, 2020].

- 1 Conservative SPDE as fluctuating continuum models
- 2 Two ways to the LDP, the skeleton equation

## The zero range process

(could also consider simple exclusion, independent particles).

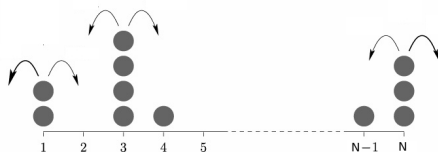


Figure: Harris, Rákos, Schütz; 2005

- State space  $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N}$ , i.e. configurations  $\eta : \mathbb{T}_N \rightarrow \mathbb{N}_0$  : System in state  $\eta$  if container  $k$  contains  $\eta(k)$  particles.
- Local jump rate function  $g : \mathbb{N}_0 \rightarrow \mathbb{R}_0^+$ .
- Translation invariant, asymmetric, zero mean transition probability

$$p(k, l) = p(k - l), \quad \sum_k kp(k) = 0.$$

- Markov jump process  $\eta(t)$  on  $\mathbb{M}_N$ .
- $\eta(k, t)$  = number of particles in box  $k$  at time  $t$ .

- Hydrodynamic limit? Multi-scale dynamics

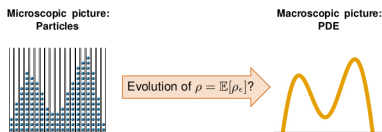


Figure: see Zimmer et. al.

- Empirical density field:  $\mu^N(x, t) := \frac{1}{N} \sum_k \delta_{\frac{k}{N}}(x) \eta(k, tN^2)$ .
- [Hydrodynamic limit - Ferrari, Presutti, Vares; 1987]

$$\mu^N(t) \rightharpoonup^* \bar{\rho}(t) dx$$

with

$$\partial_t \bar{\rho} = \frac{1}{2} \partial_{xx} \Phi(\bar{\rho})$$

with  $\Phi$  the mean local jump rate  $\Phi(\rho) = \mathbb{E}_{v_\rho} [g(\eta(0))]$ .

- Loss of information:
  - Error:  $\mu^N = \bar{\rho} + o(1)$
  - Fluctuations, rare events?

## Rate of convergence?

- Higher order expansion / fluctuation correction: Ansatz

$$\mu^N = \bar{\rho} + \frac{1}{N^{\frac{1}{2}}} Y^1 + \frac{1}{N} Y^2 + \dots$$

What are  $Y^i$ ?

- [Central limit fluctuations in non-equilibrium - Ferrari, Presutti, Vares; 1988]:  
Fluctuation density fields

$$\begin{aligned} Y^{1,N}(x,t) &= N^{\frac{1}{2}}(\mu^N(x,t) - \mathbb{E}\mu^N(x,t)) \\ &\approx N^{\frac{1}{2}}(\mu^N(x,t) - \bar{\rho}) \end{aligned}$$

Then,

$$\mathcal{L}(Y^{1,N}) \xrightarrow{*} \mathcal{L}(Y^1) \text{ for } N \rightarrow \infty$$

with  $Y^1$  the (Gaussian) solution to

$$dY^1(x,t) = \partial_{xx}(\Phi'(\bar{\rho}(x,t))Y^1(x,t))dt + \partial_x(\Phi^{\frac{1}{2}}(\bar{\rho}(x,t))dW(t))$$

with  $dW$  space-time white noise.

- Therefore,

$$\mu^N = \bar{\rho} + \overbrace{\frac{1}{N^{\frac{1}{2}}} Y^1}^{=: \bar{\rho}^N} + o\left(\frac{1}{N^{\frac{1}{2}}}\right).$$

## Rare events?

- [Large deviation principle, Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995]: Let now  $\rho_0$  constant. Then, informally,

$$\mathbb{P}[\mu^N \approx \rho dx] \approx \exp\{-N I_0(\rho dx)\},$$

with rate function

$$I_0(\rho dx) = \inf \left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dx ds : g \in L^2_{t,x}, \underbrace{\partial_t \rho = \partial_{xx} \Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)g)}_{\text{"skeleton equation"}} \right\}.$$

- Note: This does **not** coincide with the rate function of the linearly corrected continuum model  $\bar{\rho}^N$ .

**Question:** Fluctuation correction implying higher order of approximation and correct rare events?

**Ansatz:** Langevin dynamics (nonlinear!)

$$\partial_t \rho^N = \partial_{xx} \left( \Phi(\rho^N) \right) + \frac{1}{N^{\frac{1}{2}}} \partial_x \left( \Phi^{\frac{1}{2}}(\rho^N) dW_t \right).$$

Model case: Dean-Kawasaki, independent particles,  $\Phi(\rho) = \rho$ , i.e.

$$\partial_t \rho^N = \partial_{xx} \rho^N + \frac{1}{N^{\frac{1}{2}}} \partial_x \left( (\rho^N)^{\frac{1}{2}} dW_t \right).$$

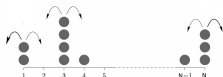
**Informal justification:**

- ① Physics: Fluctuation-dissipation relation, “fluctuating hydrodynamics”
- ② Law of large numbers, Central limit fluctuations (improved order of approximation)  
& *correct large deviations*

- Obstacle

$$\partial_t \rho = \partial_{xx} (\Phi(\rho)) + \frac{1}{N^{\frac{1}{2}}} \partial_x \left( \Phi^{\frac{1}{2}}(\rho) dW_t \right)$$

- ① not well-posed, supercritical  $\rightarrow$  no regularity structures
  - ② Renormalization? Does renormalization appear in rate function? E.g. compare  $\Phi_{2/3}^4$  [Hairer, Weber; 2014].
- Decorrelation length of discrete system =  $\frac{1}{N}$



- Ansatz: joint limit “small noise, ultraviolet cutoff”

$$\partial_t \rho^{N,K} = \partial_{xx} \left( \Phi(\rho^{N,K}) \right) + \frac{1}{\sqrt{N}} \partial_x \left( \Phi^{\frac{1}{2}}(\rho^{N,K}) \circ dW_t^K \right)$$

where  $W^K = \sum_{k=1}^K e_k \beta^k$  is a spectral (smooth) approximation of  $W = \sum_{k=1}^{\infty} e_k \beta^k$ .

- Gives the correct rate function for  $\frac{1}{N} \ll \frac{1}{K}$ .

**Note:** This is a particular case in which the link between *Macroscopic fluctuation theory* [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015] and *fluctuating hydrodynamics* [Landau-Lifshitz 1973, Spohn 1991] can be made rigorous.



## Two ways to the LDP, the skeleton equation

- 1 Conservative SPDE as fluctuating continuum models
- 2 Two ways to the LDP, the skeleton equation

- In the following concentrate on the case

$$\Phi(\rho) = \rho^m, \quad m \geq 1.$$

- We consider stochastic PDE of the type

$$\partial_t \rho^{N,K} = \Delta \left( (\rho^{N,K})^m \right) + \frac{1}{\sqrt{N}} \operatorname{div} \left( (\rho^{N,K})^{\frac{m}{2}} \circ dW_t^K \right), \quad (*)$$

on  $\mathbb{T}^d \times (0, \infty)$ , where  $W^K = \sum_{k=1}^K e_k \beta^k$ .

- Pathwise well-posedness of (\*): [Lions, Souganidis; 1998ff], [Lions, Perthame, Souganidis; 2013], [Lions, Perthame, Souganidis; 2014], [G., Souganidis; 2014], [G., Souganidis; 2015], [G., Fehrman; 2017], [Dareiotis, G.; 2019].

## Two ways to the LDP:

- 1  $\Gamma$ -convergence of the rate functional:  $N \uparrow \infty$  yields LDP for (\*) with rate function

$$I^K(\rho) = \inf \left\{ \int_0^T \int_{\mathbb{T}^d} |g|^2 dx dt : \partial_t \rho = \partial_{xx} \rho^m + \partial_x \left( \rho^{\frac{m}{2}} P^K g \right) \right\}.$$

Then consider  $K \uparrow \infty$ .

- 2 Joint scaling: Weak convergence approach to LDP ( $\frac{1}{N} \ll \frac{1}{K}$ ).

- What do we need to show? E.g.  $\Gamma$ -convergence of the rate function

$$I^K(\rho) = \inf \left\{ \int_0^T \int_{\mathbb{T}^d} |g|^2 dxdt : \partial_t \rho = \partial_{xx} \rho^m + \partial_x \left( \rho^{\frac{m}{2}} P^K g \right) \right\} \rightarrow I(\rho)$$

- Given  $\rho$  need to find recovery sequence  $\rho^K$  so that  $\rho^K \rightarrow \rho$  with

$$\limsup_K I^K(\rho^K) \leq I(\rho).$$

- Infimum is attained:  $\exists g \in L^2_{t,x}$  so that

$$I(\rho) = \int_0^T \int_{\mathbb{T}^d} |g|^2 dxdt \quad \text{and} \quad \partial_t \rho = \partial_{xx} \rho^m + \partial_x \left( \rho^{\frac{m}{2}} g \right).$$

- Choose  $g^K = P^K g$  and  $\partial_t \rho^K = \partial_{xx} (\rho^K)^m + \partial_x \left( (\rho^K)^{\frac{m}{2}} g^K \right)$ . Then

$$I^K(\rho^K) \leq \int_0^T \int_{\mathbb{T}^d} |g^K|^2 \leq \int_0^T \int_{\mathbb{T}^d} |g|^2 = I(\rho).$$

- Remains to show  $\rho^K \rightarrow \rho$ , that is, solution to

$$\partial_t \rho^K = \partial_{xx} (\rho^K)^m + \partial_x \left( (\rho^K)^{\frac{m}{2}} g^K \right)$$

converges to the (unique!) solution to

$$\partial_t \rho = \partial_{xx} \rho^m + \partial_x \left( \rho^{\frac{m}{2}} g \right).$$

- Both approaches crucially depend on understanding the skeleton PDE.
- The skeleton equation

$$\begin{aligned}\partial_t \rho &= \Delta \rho^m + \operatorname{div} \left( \rho^{\frac{m}{2}} g(t, x) \right) \\ \rho(0, x) &= \rho_0(x),\end{aligned}\tag{*}$$

with  $g \in L^2_{t,x}$ ?

- This leads to the key problem

## Problem

- 1 Existence and uniqueness of solutions to (\*).
- 2 Stability of solutions: Let  $g^n \rightharpoonup g$  in  $L^2_{t,x}$  with corresponding solutions  $\rho^n, \rho$ .  
Then

$$\rho^n \rightarrow \rho$$

in  $L^\infty_t L^1_x$ .

- Difficulty: Stable a-priori bound?  $L^p$  framework does not work.
- Do we expect non-concentration of mass / well-posedness?

## Scaling and criticality of the skeleton equation

- We consider

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d$$

with  $g \in L^q(\mathbb{R}_{+,t}; L^p(\mathbb{R}_x^d; \mathbb{R}_x^d))$  and  $\rho_0 \in L^r(\mathbb{R}_x^d)$ . E.g.

$$\partial_t \rho = \Delta \rho + \operatorname{div}(\rho^{\frac{1}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d.$$

- Via rescaling (“zooming in”):
  - $p = q = 2$  is critical.
  - $r = 1$  is critical,  $r > 1$  is supercritical.

## Apriori-bounds and energy space

- Consider

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d \quad (*)$$

with  $g \in L^2(\mathbb{R}_{+,t}; L^2(\mathbb{R}_x^d; \mathbb{R}^d))$ .

- $L^1$  estimate only gives

$$\int_{\mathbb{T}^d} \rho(t, x) dx = \int_{\mathbb{T}^d} \rho_0(x) dx.$$

- Use entropy-entropy dissipation: Evolution of entropy given by  $\int_{\mathbb{T}^d} \log(\rho) \rho$ . Informally gives

$$\int_{\mathbb{T}^d} \log(\rho) \rho dx \Big|_0^t + \int_0^t \int_{\mathbb{T}^d} (\nabla \rho^{\frac{m}{2}})^2 \lesssim \int_0^t \int_{\mathbb{T}^d} g^2.$$

- Caution: Can only be true for non-negative solutions.
- Non-standard weak solutions, rewriting (\*) as

$$\partial_t \rho = 2 \operatorname{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d,$$

e.g.  $m = 1$ ,

$$\partial_t \rho = \underbrace{2 \operatorname{div}(\rho^{\frac{1}{2}} \nabla \rho^{\frac{1}{2}})}_{=\Delta \rho} + \operatorname{div}(\rho^{\frac{1}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d,$$

- Conclusion: Have to prove uniqueness within this class of solutions.

**Ansatz for uniqueness:** Show that every weak solution is a renormalized entropy solution (extending the concepts of DiPerna-Lions, Ambrosio to nonlinear PDE). Recall DiPerna-Lions:  $\rho$  is a renormalized solution to

$$\partial_t \rho = \nabla \rho \cdot g$$

if for all  $\beta \in C^1$ ,  $\beta(\rho)$  is a solution to

$$\partial_t \beta(\rho) = \nabla \beta(\rho) \cdot g.$$

Let  $\rho$  be a weak solution to

$$\partial_t \rho = 2 \operatorname{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d.$$

Let

$$\chi(t, x, \xi) = 1_{0 < \xi < \rho(x, t)}.$$

Then, informally,

$$\partial_t \chi = m \xi^{m-1} \Delta_x \chi - g(x, t) (\partial_\xi \xi^{\frac{m}{2}}) \nabla_x \chi + (\nabla_x g(x, t)) \xi^{\frac{m}{2}} \partial_\xi \chi + \partial_\xi \rho$$

with  $\rho$  parabolic defect measure

$$\rho = \delta(\xi - \rho) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2.$$

- How to make that rigorous? Take convolution

$$\rho^\varepsilon = \varphi^\varepsilon *_x \rho.$$

- Commutator errors,

$$\begin{aligned} \partial_t \rho^\varepsilon &= \varphi^\varepsilon * \partial_t \rho = \varphi^\varepsilon * (\Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g)) \\ &= \Delta(\varphi^\varepsilon * \rho^m) + \operatorname{div}(\varphi^\varepsilon * (\rho^{\frac{m}{2}} g)) \\ &= \Delta(\rho^\varepsilon)^m + \operatorname{div}((\rho^\varepsilon)^{\frac{m}{2}} g) \\ &\quad + \Delta(\varphi^\varepsilon * \rho^m) - \Delta(\rho^\varepsilon)^m \\ &\quad + \operatorname{div}((\varphi^\varepsilon * \rho^{\frac{m}{2}}) g) - \operatorname{div}((\rho^\varepsilon)^{\frac{m}{2}} g) \\ &\quad + \operatorname{div}(\varphi^\varepsilon * (\rho^{\frac{m}{2}} g)) - \operatorname{div}((\varphi^\varepsilon * \rho^{\frac{m}{2}}) g). \end{aligned}$$

- Note: Additional commutator errors by commuting convolution and nonlinearities!
- Commutator estimate using non-standard (optimal) regularity  $\rho^{\frac{m}{2}} \in L_t^2 \dot{H}_x^1$
- Additional renormalization step to compensate low time integrability  $\rho^{\frac{m}{2}} g \in L_t^1 L_x^1$ .



**Ansatz for uniqueness:** Show that every weak solution is a renormalized entropy solution (extending the concepts of DiPerna-Lions, Ambrosio to nonlinear PDE).

### Theorem

A function  $\rho \in L_t^\infty L_x^1$  is a weak solution to

$$\partial_t \rho = 2 \operatorname{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \operatorname{div}(\rho^{\frac{m}{2}} g)$$

if and only if  $\rho$  is a renormalized entropy solution.

### Uniqueness for renormalized entropy solutions (variable doubling)

- Additional errors from space-inhomogeneity (with little regularity)
- Note: Entropy dissipation measure

$$q(x, \xi, t) = \delta(\xi - \rho(x, t)) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2$$

does not satisfy

$$\lim_{|\xi| \rightarrow \infty} \int_{t,x} q(x, \xi, t) dx dt = 0.$$

- Established arguments [Chen, Perthame; 2003] not applicable.

## Theorem (The skeleton equation)

Let  $g \in L^2([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$ ,  $\rho_0 \in L^1(\mathbb{T}^d)$  non-negative and  $\int \rho_0 \log(\rho_0) dx < \infty$ ,  $m \in [1, \infty)$ .

1 There is a unique weak solution

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d. \quad (*)$$

For two weak solutions  $\rho^1, \rho^2 \in L^\infty([0, T]; L^1(\mathbb{T}^d))$  we have

$$\|\rho^1 - \rho^2\|_{L^\infty([0, T]; L^1(\mathbb{T}^d))} \leq \|\rho_0^1 - \rho_0^2\|_{L^1(\mathbb{T}^d)}.$$

2 Let  $\{g_n\}_{n \in \mathbb{N}} \subseteq L^2([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$  with

$$\lim_{n \rightarrow \infty} g_n = g \text{ weakly in } L^2([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$$

and let  $\rho_n \in L^1([0, T]; L^1(\mathbb{T}^d))$  be the corresponding solutions with control  $g_n$ . Then,

$$\lim_{n \rightarrow \infty} \rho_n = \rho \text{ strongly in } L^1([0, T]; L^1(\mathbb{T}^d))$$

where  $\rho \in L^1([0, T]; L^1(\mathbb{T}^d))$  is the solution with control  $g$ .

Consider

$$d\rho^N = \Delta(\rho^N)^m dt + \frac{1}{\sqrt{N}} \operatorname{div} \left( (\rho^N)^{\frac{m}{2}} \circ dW^{K(N)}(t) \right).$$

Theorem (Large deviation principle)

Let  $K(N)$ ,  $n(N) \rightarrow \infty$  with  $\frac{K(N)^3}{N} \rightarrow 0$  for  $N \rightarrow \infty$ . For  $\rho_0 \in L^{m+1}(\mathbb{T}^d)$  and  $\rho \in L^\infty([0, T]; L^1(\mathbb{T}^d))$  let

$$I_{\rho_0}(\rho) := \inf \left\{ \frac{1}{2} \int_0^T \|g(s)\|_{L_x^2}^2 ds : g \in L_{t,x}^2, \partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \right\}.$$

Then, the family  $\{\rho^N\}$  satisfies the large deviation principle on  $L^\infty([0, T]; L^1(\mathbb{T}^d))$  with good rate function  $I_{\rho_0}$ , uniformly on compact subsets of  $L^{m+1}(\mathbb{T}^d)$ .



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