# Generation of random dynamical systems for SPDE with non linear noise

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joint work with: Ben Fehrman
[G., Fehrman; arxiv, 2017].

Outline:

1. Introduction: Generation of RDS by SPDE
2. Nonlinear diffusion equations with conservative, nonlinear noise
3. Aspects of the proof

## Generation of RDS by SPDE

- Aim: Generation of random dynamical systems from stochastic (partial) differential equations
- Motivation: Application of methods from dynamical systems (multiplicative ergodic theorem, invariant manifolds, Lyapunov exponents) to stochastic differential equations.

- Consider SDE

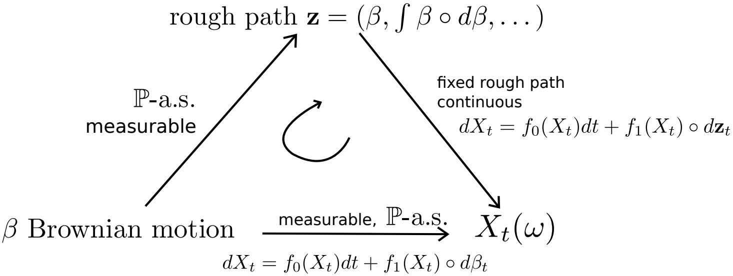
$$
\begin{aligned}
dX_t^x(\omega) &= f_0(X_t^x(\omega))dt + f_1(X_t^x)\circ d\beta_t(\omega) \\
X_0^x &= x \in H.
\end{aligned}
\tag{1}
$$

- Obstacle: $X_t^x$ solves (1) for each $x$, $\mathbb{P}$-a.s., no $\omega$-wise solution.
- Flow property

$$
\varphi(t,s;\omega) = \varphi(t,r;\omega)\circ\varphi(r,s;\omega), \quad \forall s \le r \le t,
$$

  with $X_t^x(\omega) := \varphi(t,s;\omega)x$ solves the stochastic equation with $X_s^x(\omega) = x$.
- Solution: $dim(H) < \infty$
  - Kolmogorov continuity theorem to prove continuous dependence on $x$.
  - Stochastic equations can be 'lifted' to equations of homeomorphisms, i.e. solved for all $x$ simultaneously.

- Alternative: rough path theory




- Immediate consequence: Generation of a stochastic flow
- Restriction: Requires smooth coefficients $f_0$, $f_1$.
- $\infty$-dimensional case:
  - Kolmogorov continuity theorem does not apply
  - Do not expect flow of homeomorphisms
  - No general rough path theory (coefficients lacking regularity)

Transformation method: Affine linear noise

- E.g. consider
$$du = A(u)dt + dW_t^1 + udW_t^2 + \nabla u \circ dW_t^3.$$
- SPDE can be transformed into a PDE with random coefficients.
- As long as random PDE well-posed: Generation of an RDS follows.
- How to deal with non-linear noise? E.g. [Flandoli, 1995].

- [Diehl, Friz, AoP, 2012]

$$du = Lu\,dt + f(u,\nabla u)dt + H(x,u)\circ d\beta_t,$$

  $L$ a linear, uniformly elliptic operator.
- Path-by-path treatment based on BSDE representation and rough path analysis.
- Generation of an RDS is an immediate consequence.
- **Open Problem:** RDS for

$$\partial_t u = \Delta u + \nabla\cdot(A(x,u)\circ d\beta_t),$$

  where

$$\nabla\cdot(A(x,u)\circ d\beta_t) = \sum_{i,j}\partial_{x_i}(A^{i,j}(x,u)\circ d\beta_t^j).$$

## Nonlinear diffusion equations with conservative, nonlinear noise

- We consider

$$\begin{aligned} \partial_t u &= \Delta(|u|^{m-1}u) + \nabla \cdot (A(x,u) \circ d\boldsymbol{z}_t) \quad &&\text{on } \mathbb{T}^d \times (0,\infty), \\ u &= u_0 &&\text{on } \mathbb{T}^d \times \{0\}, \end{aligned}$$

  for $d \geq 1$, $m \in (0,\infty)$.
- In particular

$$\partial_t u = \Delta u + \operatorname{div} f(x,u) + \nabla \cdot (A(x,u) \circ d\beta_t).$$

- Applications:
  - Fluctuating hydrodynamics for zero range process
  - Limits of weakly interacting diffusions (mean field games)
  - Dean-Kawasaki model (passive scalars in turbulent fluid with thermal noise)
  - Thin film equations

- Recall

$$\begin{aligned} \partial_t u &= \Delta(|u|^{m-1}u) + \nabla \cdot (A(x,u) \circ d\boldsymbol{z}_t) && \text{on } \mathbb{T}^d \times (0,\infty), \\ u &= u_0 && \text{on } \mathbb{T}^d \times \{0\}, \end{aligned}$$

  for $m \in (0,\infty)$.
- Assumptions:
  - Driving noise: For some $n \geq 1$, $\alpha \in (0,1)$,

$$z_t = (z_t^1, \dots, z_t^n) \in C^{0,\alpha}\left([0,T]; G^{\lfloor \frac{1}{\alpha} \rfloor}(\mathbb{R}^n)\right).$$

  - Regularity of the coefficients: For $\gamma > \frac{1}{\alpha}$,

$$\nabla_x A(x,v) \in C^{\gamma+2}(\mathbb{T}^d \times \mathbb{R}),\ \partial_v A(x,v) \in C^{\gamma+2}(\mathbb{T}^d \times \mathbb{R}).$$

  - No source:

$$\nabla_x \cdot A^t(x,0) = 0 \in \mathbb{R}^n \text{ for each } x \in \mathbb{T}^d.$$

### Theorem

*Let* $u_0^1, u_0^2 \in L^2_+(\mathbb{T}^d)$ *and* $u^1$ *and* $u^2$ *be entropy solutions. Then*

$$\left\|u^1 - u^2\right\|_{L^\infty_t([0,\infty);L^1_x(\mathbb{T}^d))} \le \left\|u_0^1 - u_0^2\right\|_{L^1(\mathbb{T}^d)}.$$

*In particular, entropy solutions are unique.*

### Theorem

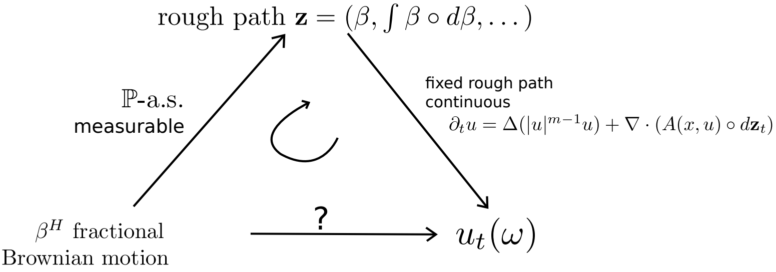
*Let* $u_0 \in L^2_+(\mathbb{T}^d)$*. There exists a unique non-negative entropy solution with initial data* $u_0$*. Furthermore,*

$$\left\|u\right\|_{L^\infty_t([0,\infty);L^1_x(\mathbb{T}^d))} \le \left\|u_0\right\|_{L^1(\mathbb{T}^d)}.$$

Extensions:

- For $m > 2$ or $m = 1$, non-negativity of $u_0$ can be avoided.
- For $m \ge 3$ or $m = 1$, $u_0 \in (L^1 \cap L^2)(\mathbb{R}^d)$ the Cauchy problem can be treated by identical methods.
- Integrability: Localization allows extension to $L^1$-data.

Application to fractional Brownian motion:




### Theorem

*Let $t \in [0,\infty) \mapsto z_t(\omega)$ be the sample paths of a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, 1)$ on a probability space $\omega \in (\Omega, \mathscr{F}, \mathbb{P})$. Then $u$ defines a (continuous) random dynamical system on $L^2_+(\mathbb{T}^d)$.*

# Aspects of the proof

- Consider

$$\begin{cases} \partial_t u = \Delta(|u|^{m-1}u) + \nabla \cdot (A(x,u) \circ d\mathbf{z}_t) & \text{on } \mathbb{T}^d \times (0,\infty), \\ u = u_0 & \text{on } \mathbb{T}^d \times \{0\}, \end{cases}$$

  for $m \in (0,\infty)$.
- Kinetic formulation: Let

$$\chi(t,x,v) := 1_{v<u(t,x)} - 1_{v<0}.$$

  Then

$$\partial_t \chi = m|v|^{m-1} \Delta_x \chi + \nabla_x \chi (\partial_v A(x,v) \circ d\mathbf{z}_t) - \partial_v \chi \left(\nabla_x \cdot A^t(x,v) \circ d\mathbf{z}_t\right) + \partial_v q$$

  for some non-negative measure $q$.
- Random test-functions (duality method) inspired by stochastic viscosity solutions.

- Recall: Kinetic formulation

$$\partial_t \chi = m|v|^{m-1}\Delta_x \chi + \nabla_x \chi(\partial_v A(x,v)\circ d\boldsymbol{z}_t) - \partial_v \chi\left(\nabla_x \cdot A^t(x,v)\circ d\boldsymbol{z}_t\right) + \partial_v q$$

- Consider, for each $t_0, t_1 \in [0,\infty)$ and $\rho_0 \in C_c^\infty(\mathbb{T}^d \times \mathbb{R})$,

$$\begin{aligned}\partial_t \rho_{t_0,t} &= (\partial_v A(x,v)\circ d\boldsymbol{z}_t)\cdot \nabla_x \rho_{t_0,t} - (\nabla_x \cdot A^t(x,v)\circ d\boldsymbol{z}_t)\,\partial_v \rho_{t_0,t}\\ \rho_{t_0,t_0} &= \rho_0.\end{aligned}$$

- Then

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{T}^d} \chi(x,v,s)\rho_{t_0,s}(x,v)\,\mathrm{d}x dv\Big|_{s=t_0}^{t_1}\\ &= \int_{t_0}^{t_1}\int_{\mathbb{R}}\int_{\mathbb{T}^d}\left(m|v|^{m-1}\right)\chi(x,v,s)\Delta_x \rho_{t_0,s}(x,v)\,\mathrm{d}x dv\,\mathrm{d}s\\ &- \int_{t_0}^{t_1}\int_{\mathbb{R}}\int_{\mathbb{T}^d} q(x,\xi,s)\partial_v \rho_{t_0,s}(x,v)\,\mathrm{d}x dv\,\mathrm{d}s.\end{aligned}$$

- This gives a stable form of the SPDE.

- Recall

$$\partial_t \rho_{t_0,t} = (\partial_v A(x,v) \circ d\mathbf{z}_t) \cdot \nabla_x \rho_{t_0,t} - (\nabla_x \cdot A^t(x,v) \circ d\mathbf{z}_t)\, \partial_v \rho_{t_0,t}$$
$$\rho_{t_0,t_0} = \rho_0.$$

- Characteristics:

$$\begin{aligned} \mathrm{d}Y^{x,v}_{t_0,t} &= \partial_v A(Y^{x,v}_{t_0,t}, \Pi^{x,v}_{t_0,t}) \circ d\mathbf{z}_{t_0,t} && \text{in } (0,t_0), \\ \mathrm{d}\Pi^{x,v}_{t_0,t} &= -\nabla_x \cdot A^t(Y^{x,v}_{t_0,t}, \Pi^{x,v}_{t_0,t}) \circ d\mathbf{z}_{t_0,t} && \text{in } (0,t_0), \\ (Y^{x,v}_{t_0,0}, \Pi^{x,v}_{t_0,0}) &= (x,v). \end{aligned}$$

- Solve the system of characteristics by rough path methods.
- Then

$$\rho_{t_0,t}(x,v) = \rho_0\left(Y^{x,v}_{t,t-t_0}, \Pi^{x,v}_{t,t-t_0}\right).$$

- Uniqueness of entropy solutions
  - Commutator estimates
  - To control errors: Exploit new cancellations
  - Use delicate (and new) regularity estimates on $u$.