Stochastic thin film equations

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joint work with: Manuel Gnann.
[G., Gnann; arxiv, 2019].

- Consider (thin) film of fluid on a 1d-substrate

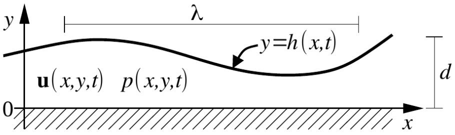

Figure 1: A thin liquid film on a flat one-dimensional substrate (coinciding with the x-axis). The film surface (i.e., the moving boundary) is parameterized by the film thickness $h(x,t)$. The flow is characterized by the flow velocity $\mathbf{u} = (u_x, u_y)$ and the pressure p.

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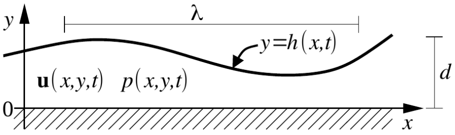

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- Dynamics of the fluid: Navier-Stokes equations

$$\begin{aligned} \operatorname{div} u &= 0 \\ \rho \partial_t u &= \eta \Delta u - \rho (u \cdot \nabla) u - \nabla p. \end{aligned}$$

Boundary conditions:

- fluid-solid interface: No slip

$$u = 0 \text{ at } y = 0.$$

- fluid-air interface: Stress - curvature (surface tension) balance

$$\sigma \cdot n = \gamma \kappa n,$$

where σ is the stress tensor

$$\sigma_{ij} = \eta(\partial_i u_j + \partial_j u_i) - p\delta_{ij},$$

κ is the mean curvature, and n is the outer normal

$$n = (-\partial_x h, 1)/\sqrt{(1 + (\partial_x h)^2)}.$$

- Triple boundary fluid-solid-air:

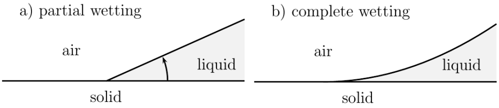

Figure 1.3: Wetting regimes

Lubrication approximation:

- Aim: Write a PDE for the evolution of the height h of the thin fluid film.
- Consider a fluid particle at the fluid-air interface

$$(x(t), h(t, x(t)))$$

- Change of height of the fluid

$$\frac{d}{dt} h(t, x(t)) = u_y(x(t), h(t, x(t)))$$

- Since the particle moves with the fluid

$$\dot{x}(t) = u_x(x(t), h(t, x(t)))$$

- Chain rule gives

$$\begin{aligned}\frac{d}{dt} h(t, x(t)) &= (\partial_t h)(t, x(t)) + (\partial_x h)(t, x(t))\dot{x}(t) \\ &= (\partial_t h)(t, x(t)) + (\partial_x h)(t, x(t)) u_x(x(t), h(t, x(t)))\end{aligned}$$

- Hence,

$$\partial_t h = u_y - (\partial_x h) u_x$$

on the fluid-air interface.

- Thin film equation:

$$\partial_t h = -\partial_x (h^3 \partial_{xxx} h).$$

with a free interface at $\partial\{h > 0\}$.

- Different fluid-solid boundary conditions

$$\partial_t h = -\partial_x (h^n \partial_{xxx} h),$$

with $n \in [1, 3]$.

Note:

- Degenerate PDE
- Forth order PDE $\rightarrow$ no comparison arguments
- No complete well-posedness theory in the deterministic case (!)

Limits of the deterministic thin film equation

Introduction of stochastic thin film equations is motivated from three angles:

1. Wrong prediction of rupture time scale
2. Fluctuations in small scales
3. Front propagation

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2. Fluctuations in small scales
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Rupture time scales [Grün, Mecke, Rauscher; J. Stat. Phys. 2006]

- Dynamics: Rupture time-scale and droplet formation time-scale

Figure 2: Snapshots at times $t = 10, 12, 13, 14, 15, 15.5, 15.75, 16$ for the deterministic thin-film equation. Discretisation parameters are the grid spacing $a = 2^{-9}\,L$ and time step $\tau = a^{\frac{5}{2}}$.

- Experiments: Deterministic thin film predicts the quotient

$$\frac{\text{rupture time-scale}}{\text{droplet-formation time-scale}}$$

too large compared to experiments.

Fluctuations in small scales [Davidovitch, Moro, Stone; Phys. Rev. Let. 2005]

- In very thin films thermal fluctuations are relevant: Hydrodynamic description possibly not justified and thermic fluctuations are present at small scales.

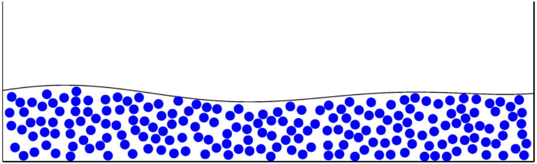

FIG. 1. *A thin liquid film consisting only of a limited number of fluid molecules.*

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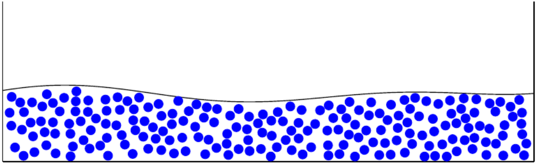

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Front propagation

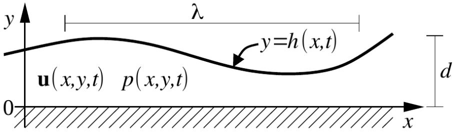
- Conjecture: For

$$\partial_t h = -\partial_x (h^n \partial_{xxx} h),$$

  with $n \geq 3$ the free interface $\partial\{\operatorname{supp} h(t)\}$ is constant.
- Based on the non-existence of self-similar solutions for $n \geq 3$.
- Proven only for $n \geq \frac{7}{2}$.

Derivation of the stochastic thin film equation

- How should the noise in a *stochastic* thin film equation look like?
- Consider (thin) film of fluid on a 1d-substrate



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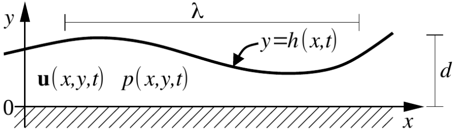

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- Dynamics of the fluid: Stochastic Navier-Stokes equations [Landau-Lifschitz; Vol 2]

$$\operatorname{div} u = 0$$
$$\rho \partial_t u = \eta \Delta u - \rho (u \cdot \nabla) u - \nabla p + \operatorname{div} S,$$

where S is space time white noise.

- Following the Lubrication approximation argument leads to the stochastic thin film equation
$$\partial_t h = -\partial_x(h^3 \partial_{xxx} h) + \partial_x(h^{\frac{3}{2}} \xi),$$
where ξ is space-time white noise.
- Different fluid-solid boundary conditions:
$$\partial_t h = -\partial_x(h^n \partial_{xxx} h) + \partial_x(h^{\frac{n}{2}} \xi),$$
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with $n \in [1,3]$.
- But: How should the stochastic integral be interpreted?

Recall: First step in derivation of the thin film equation

- Consider a fluid particle at the fluid-air interface

$$(x(t), h(t, x(t)))$$

- Change of height of the fluid

$$\frac{d}{dt} h(t, x(t)) = u_y(x(t), h(t, x(t)))$$

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- This suggests (informally) Stratonovich noise!

Known result(s)

- [Fischer, Grün, SIMA 2018]: Consider the stochastic thin film equation with interface potential Φ

$$\partial_t h = -\partial_x(h^2\partial_{xxx}h) + \partial_x(h^2\partial_x\Phi'(h)) + \partial_x(h\,dW).$$

- Note:
  - quadratic mobility: $n = 2$
  - Interface potential Φ assumed to satisfy (!)

$$c_1 h^{-p-2} - c_2 \leq \Phi''(h) \leq C_1 h^{-p-2}$$
$$Ch^{-p} \leq \Phi(h)$$

    for some $p > 2$.
  - Strictly positive initial data (!)
  - Noise W spatially smooth enough
- Result: (Probabilistically weak) Existence of a solution.

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- Result: (Probabilistically weak) Existence of a solution.
- Aims in the following:
  1. Construct solutions for non-negative initial data (e.g. compactly supported)
  2. Do not assume presence of interface potential
  3. Construct via simple to implement numerical method

Main result and idea of construction

- Stochastic thin film equation with quadratic mobility: $n = 2$, i.e.

$$\partial_t h = -\partial_x(h^2 \partial_{xxx} h) + \partial_x(h \circ dW) \qquad (\star)$$

  on the torus $\mathbb{T}$.
- Write

$$W_t = \sum_k \lambda_k \psi_k \beta^k$$

  with ψ_k eigenvectors of the Laplacian in $H^2(\mathbb{T})$ and assume

$$\sum_k \lambda_k^2 < \infty.$$

Theorem (G., Gnann; 2019)

Let $h_0 \in H^1(\mathbb{T})$ non-negative. Then, there is a martingale solution h to $(\star)$ such that $h \geq 0$ almost surely and, for all $p \geq 2$,

$$\mathbb{E} ess\ sup_{t\in[0,T]} \|h(t)\|^p_{W^{1,2}} \leq C \|h_0\|^p_{W^{1,2}}.$$

- Trotter-Kato scheme:
    - Split the time interval $[0,T]$ into pieces $[(j-1)\delta, j\delta)$, with $\delta = \frac{T}{N+1}$.
    - On each interval: First let the deterministic dynamics run, then let the stochastic dynamics run.

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  - Split the time interval $[0,T]$ into pieces $[(j-1)\delta, j\delta)$, with $\delta = \frac{T}{N+1}$.
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- Precisely:
  - Deterministic dynamics: On $[(j-1)\delta, j\delta)$, v_N satisfies

$$\partial_t v_N = -\partial_x(v_N^2 \partial_{xxx} v_N).$$

  - Stochastic dynamics: On $[(j-1)\delta, j\delta)$, w_N satisfies

$$dw_N = \partial_x(w_N \circ dW).$$

    Meaning: $W_t = \sum \lambda_k \psi_k \beta^k$ and

$$dw_N = \frac{1}{2}\sum_{k\in\mathbb{Z}} \lambda_k^2 \partial_x(\psi_k \partial_x(\psi_k w_N)) + \sum_{k\in\mathbb{Z}} \lambda_k \partial_x(w_N \psi_k) d\beta^k.$$

  - Glueing together:

$$v_N(0) := h_0, \quad v_N(j\delta) := \lim_{t\uparrow j\delta} w_N(t), \quad w_N((j-1)\delta) := \lim_{t\uparrow (j-1)\delta} v_N(t).$$

- Note: Simple implementation re-using the existing deterministic codes.

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$$
\begin{aligned}
v_N(t) - h_0 \quad &= \quad v_N(t) + \sum_{j=1}^{\lfloor \frac{t}{\delta} \rfloor} \big(- v_N(j\delta) + w_N(j\delta - 0) \big) \\
&\quad + \sum_{j=1}^{\lfloor \frac{t}{\delta} \rfloor} \big(v_N(j\delta - 0, \cdot) - w_N((j-1)\delta) \big) - v_N(0) \\
&= \quad - \int_0^t \partial_x (v_N^2 (\partial_{xxx} v_N))\, ds \\
&\quad + \frac{1}{2} \sum_{k \in \mathbb{Z}} \lambda_k^2 \int_0^{\lfloor \frac{t}{\delta} \rfloor \delta} \partial_x \left(\psi_k \partial_x \left(\psi_k w_N \right) \right) ds \\
&\quad + \sum_{k \in \mathbb{Z}} \lambda_k \int_0^{\lfloor \frac{t}{\delta} \rfloor \delta} \partial_x \left(\psi_k w_N \right) d\beta^k(s).
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\end{aligned}
$$

- To pass to the limit $N \to \infty$ we need uniform a-priori estimates.
- I.e. aim to prove tightness of the laws $\mathscr{L}(v_N)$.

Energy estimate:

$$\partial_t v = -\partial_x(v^2 \partial_{xxx} v).$$

Multiply by $-\partial_{xx} v$ and integrate to get

$$\partial_t \|\partial_x v\|_2^2 = -\int v^2 (\partial_{xxx} v)^2.$$

Corollary

Let $v_0 \in H^1(\mathbb{T})$. There exists a solution v such that, for $p \in [2,\infty)$,

$$\|\partial_x v(t,\cdot)\|_2^p + 2\int_0^t \|\partial_x v(s,\cdot)\|_2^{p-2} \int_{\{v(t',\cdot)>0\}} (v(s,x))^2 (\partial_{xxx} v(s,x))^2 \, dx \, ds \le \|\partial_x v_0\|_2^p.$$

Deterministic dynamics:

Theorem (Beretta, Bertsch, Dal Passo; 1995; Bertozzi, Pugh; 1996)

Assume that $v_0 \in H^1(\mathbb{T})$ *with* $v_0 \geq 0$. *Then, there exists a function* $v\colon [0,\delta) \times \mathbb{T} \to [0,\infty)$ *satisfying*

- $v \in C^{\frac{1}{8},\frac{1}{2}}([0,\delta] \times \mathbb{T}) \cap L^\infty([0,\delta); H^1(\mathbb{T}))$
- $v^2 \partial_x^3 v \in L^2(\{v > 0\})$.
- *Mass conservation:* $\int_0^L v\,dx = \int_0^L v_0\,dx$ *on the time interval* $[0,\delta)$.
- *The function* v *satisfies, for all* $\phi \in C_c^\infty((0,\delta); C^\infty(\mathbb{T}))$,

$$\int_0^\delta \int_0^L v\,(\partial_t \phi)\,dx\,dt + \int_0^\delta \int_{\{v(t,\cdot)>0\}} v^2\,(\partial_x^3 v)\,(\partial_x \phi)\,dx\,dt = 0 \tag{1}$$

Initial value: $v(0,\cdot) = v_0$ *in the sense that* $\|v(t,\cdot) - v_0\|_{W^{1,2}} \to 0$ *as* $t \searrow 0$.

Stochastic dynamics:

Proposition

Let $p \in [2,\infty)$, $w_0 \in L^p\left(\Omega; H^1(\mathbb{T})\right)$. Then

- *There exists a solution w to*

$$dw = \partial_x (w \circ dW).$$

with initial data w_0 satisfying

$$\mathbb{E} ess\, sup_{t\in[0,\delta)} \|w(t,\cdot)\|^p_{W^{1,2}} \leq C_1\, \mathbb{E}\|w_0\|^p_{W^{1,2}},$$

$$\limsup_{t\nearrow\delta} \mathbb{E}\|\partial_x w(t,\cdot)\|_2^p \leq e^{C_2\delta}\left(\mathbb{E}\|\partial_x w_0\|_2^p + C_3\,\delta\,\mathbb{E}\left|\int_0^L w_0\, dx\right|^p\right),$$

where $C_1, C_2, C_3 < \infty$ are independent of δ, w, and w_0.

- *Mass is conserved: $\int_0^L w(t,\cdot)\, dx = \int_0^L w_0\, dx$ for $t \in [0,\delta)$, $\mathbb{P}$-almost surely.*
- *Nonnegativity: If $w_0 \geq 0$, then $w \geq 0$, $\mathbb{P}$-almost surely.*

Note: Stochastic coercivity [Krylov-Rozovskii, Pardoux, Liu-Röckner] not satisfied.

Compose dynamics in a single function:

$$h_N(t,\cdot) := \begin{cases} v_N(2t-(j-1)\delta,\cdot) & \text{for} \quad t \in \left[(j-1)\delta,(j-\tfrac{1}{2})\delta\right), \\ w_N(2t-j\delta,\cdot) & \text{for} \quad t \in \left[(j-\tfrac{1}{2})\delta,j\delta\right), \end{cases}$$

where $j \in \{1,\ldots,N+1\}$.

Proposition

There exists a constant $C < \infty$ such that for all $N \in \mathbb{N}$ we have

$$v_N, w_N \in L^p\left(\Omega; L^\infty\left([0,T); H^1(\mathbb{T})\right)\right)$$

with

$$\mathbb{E} \sup_{t\in[0,T)} \|h_N(t,\cdot)\|_{W^{1,2}}^p + \mathbb{E} \sup_{t\in[0,T)} \|v_N(t,\cdot)\|_{W^{1,2}}^p + \mathbb{E} \sup_{t\in[0,T)} \|w_N(t,\cdot)\|_{W^{1,2}}^p$$
$$+ \mathbb{E} \int_0^T \|v_N(t,\cdot)\|_{W^{1,2}}^{p-2} \int_{\{v_N(t,\cdot)>0\}} (v_N \partial_x^3 v_N)^2 \, dx \, dt \le C \|u_0\|_{W^{1,2}}^p.$$

Caution: Have to control that constants do not explode due to iteration of the estimates (!).

Recall:

$$dv_N = \partial_x(v_N^2 \partial_{xxx} v_N),$$
$$dw_N = \partial_x(w_N \circ dW) = \frac{1}{2}\sum_{k\in\mathbb{Z}} \lambda_k^2 \partial_x(\psi_k \partial_x(\psi_k w_N)) + \sum_{k\in\mathbb{Z}} \lambda_k \partial_x(w_N \psi_k) d\beta^k.$$

This allows to deduce regularity in time from regularity in space. E.g.

$$v_N \in C^{\frac{1}{2}}([0,T); W^{-1,2}(\mathbb{T}))$$
$$w_N = w_N^1 + w_N^2 \in C^{1-}([0,T); W^{-1,2}(\mathbb{T})) + W^{\frac{1}{2}-,\infty-}([0,T); L^2(\mathbb{T})).$$

Note:

- Typical setup of application of Aubin-Lions-Simon's Lemma
- Two estimates. First all regularity in space component, second all regularity in time components -> need anisotropic interpolation

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Lemma

Suppose $\delta \in (0,\infty)$, $r_1, r_2, s_1, s_2 \in \mathbb{R}$, $q \in [1,\infty)$, and $\kappa \in [0,1]$. Then

$$\left(B_q^{r_1,q}\left([0,\delta); B_q^{s_1,q}(\mathbb{T})\right), B_q^{r_2,q}\left([0,\delta); B_q^{s_2,q}(\mathbb{T})\right)\right)_{\kappa,q} = B_q^{r,q}\left([0,\delta); B_q^{s,q}(\mathbb{T})\right), \quad (2)$$

where $r = (1-\kappa)r_1 + \kappa r_2$ and $s = (1-\kappa)s_1 + \kappa s_2$. The norms in (2) are equivalent, with bounds that are independent of δ

Proposition (regularity in time)

For any $\varepsilon > 0$, $\kappa \in (2\varepsilon, 2p^{-1}) \cap (2\varepsilon, \frac{1}{2}]$, *and* $q \in (\frac{2}{\kappa-2\varepsilon}, \infty)$ *there exists* $C < \infty$ *such that for all* $N \in \mathbb{N}$ *we have*

$$h_N \in L^p\left(\Omega; W^{\frac{\kappa}{2}-\varepsilon,q}\left([0,T); W^{\frac{1}{2}-2\kappa,q}(\mathbb{T})\right)\right)$$

with

$$\mathbb{E}\|h_N\|^p_{W^{\frac{\kappa}{2}-\varepsilon,q}\left([0,T);W^{\frac{1}{2}-2\kappa,q}(\mathbb{T})\right)} \leq C\|h_0\|^p_{W^{1,2}}\left(1+\|h_0\|^{\kappa p}_{W^{1,2}}\right).$$

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Theorem (Jakubowski)

Suppose that $(\mathscr{X}, \mathscr{T})$ *is a topological space, countably separated. Let* $(X_N)_{N\in\mathbb{N}}$ *be a sequence of* $\mathscr{X}$*-valued random variables and that for all* $M \in \mathbb{N}$ *there exists* $\mathscr{K}_M \Subset \mathscr{X}$ *such that* $\inf_N \mathbb{P}\{X_N \in \mathscr{K}_M\} > 1 - \frac{1}{M}$ *(tightness).*
Then, there exists a subsequence of $(X_N)_{N\in\mathbb{N}}$ *and random variables* $\tilde{X}, \tilde{X}_N : [0,1] \to \mathscr{X}$, *such that* $X_N \sim \tilde{X}_N$ *and*

$$\lim_{N\to\infty} \tilde{X}_N(\omega) = \tilde{X}(\omega) \quad \textit{for all } \omega \in [0,1].$$

Proposition (point-wise convergence)

We define the spaces

$$\mathscr{X}_h := BC^0\left([0,T)\times\mathbb{T}\right),$$
$$\mathscr{X}_J := L^2\left([0,T)\times\mathbb{T}\right) \quad \textit{endowed with the weak topology},$$
$$\mathscr{X}_W := BC^0\left([0,T); H^2(\mathbb{T})\right).$$

Then, there exist random variables $\tilde{h}, \tilde{h}_N : [0,1] \to \mathscr{X}_u$, $\tilde{J}_N, \tilde{J} : [0,1] \to \mathscr{X}_J$, *and* $\tilde{W} : [0,1] \to \mathscr{X}_W$ *with*

$$\left(\tilde{h}_N, \tilde{J}_N, \tilde{W}\right) \sim (h_N, J_N, W), \quad \textit{where} \quad J_N := \mathbb{I}_{\{v_N>0\}}\, v_N^2(\partial_x^3 v_N),$$

and, as $N \to \infty$, *for subsequence,*

$$\tilde{h}_N(\omega) \to \tilde{h}(\omega) \textit{ in } \mathscr{X}_u$$
$$\tilde{J}_N(\omega) \rightharpoonup \tilde{J}(\omega) \textit{ in } \mathscr{X}_J$$

for every $\omega \in [0,1]$

Lemma

Assume that $\tilde{h}_N$, $\tilde{v}_N$, $\tilde{w}_N$, $\tilde{h}$, $\tilde{v}$, and $\tilde{w}$ are given as above. Then, for any $\varphi \in C^\infty(\mathbb{T})$ and $t \in [0,T)$, and up to taking subsequences, we have

$$(\tilde{v}_N(t,\cdot),\varphi)_2 \to \left(\tilde{h}(t,\cdot),\varphi\right)_2,$$

$$\int_0^t \int_{\{\tilde{v}_N(t',\cdot)>0\}} \tilde{v}_N^2 \left(\partial_x^3 \tilde{v}_N\right) (\partial_x \varphi)\, dx\, ds \to \int_0^t \int_{\{\tilde{u}(t',\cdot)>0\}} \tilde{h}^2 \left(\partial_x^3 \tilde{h}\right) (\partial_x \varphi)\, dx\, ds,$$

$$\sum_{k\in\mathbb{Z}} \lambda_k^2 \int_0^{\lfloor \frac{t}{\delta} \rfloor \delta} (\psi_k \partial_x (\psi_k \tilde{w}_N), \partial_x \varphi)_2\, ds \to \sum_{k\in\mathbb{Z}} \lambda_k^2 \int_0^t \left(\psi_k \partial_x \left(\psi_k \tilde{h}\right), \partial_x \varphi\right)_2 ds,$$

$$\sum_{k\in\mathbb{Z}} \lambda_k \int_0^{\lfloor \frac{t}{\delta} \rfloor \delta} \left(\psi_k \tilde{h}_N, \partial_x \varphi\right)_2 d\tilde{\beta}_N^k(s) \to \sum_{k\in\mathbb{Z}} \lambda_k \int_0^t \left(\psi_k \tilde{h}, \partial_x \varphi\right)_2 d\tilde{\beta}^k(s)$$

as $N \to \infty$, $\tilde{\mathbb{P}}$-almost surely.

Hence, we can pass to the limit in

$$\tilde{v}_N(t) - h_0 = -\int_0^t \partial_x(\tilde{v}_N^2(\partial_{xxx}\tilde{v}_N))\, ds + \frac{1}{2}\sum_{k\in\mathbb{Z}} \lambda_k^2 \int_0^{\lfloor \frac{t}{\delta} \rfloor \delta} \partial_x(\psi_k \partial_x(\psi_k \tilde{w}_N))\, ds$$

$$+ \sum_{k\in\mathbb{Z}} \lambda_k \int_0^{\lfloor \frac{t}{\delta} \rfloor \delta} \partial_x(\psi_k \tilde{w}_N)\, d\beta^k(s).$$

References

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