Large deviations for conservative, stochastic PDE and non-equilibrium fluctuations

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joint work with: Ben Fehrman.
[G., Fehrman; arxiv, 2020].

Introduction: Large deviations in zero range process

1. Introduction: Large deviations for the zero range process
   - Fluctuations in the zero range process
   - Link to stochastic PDE

2. Two ways to the LDP
   - Scaling and criticality for the skeleton equation

The zero range process
(could also consider simple exclusion, independent particles).

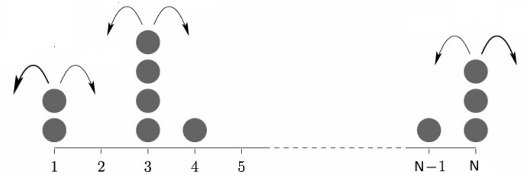

Figure: Harris, Rákos, Schütz; 2005

- Local jump rate function $g : \mathbb{N}_0 \to \mathbb{R}_0^+$.
- Translation invariant, asymmetric, zero mean transition probability

$$p(x,y) = p(x-y), \quad \sum_k kp(k) = 0.$$

- Hydrodynamic limit?

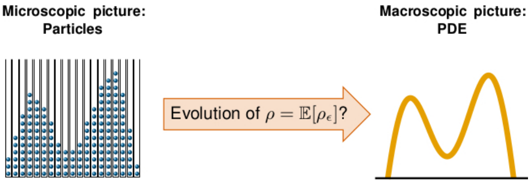

Figure: see Zimmer et. al.

- Empirical density field

$$\mu_t^N(x) := \frac{1}{N^d} \sum_k \delta_{\frac{k}{N}}(x) \eta_{tN^2}(k).$$

- [Hydrodynamic limit - Ferrari, Presutti, Vares; 1987]

$$\mu_t^N \rightharpoonup^* \bar{\rho}_t \, dx$$

with
$$\partial_t \bar{\rho} = \frac{1}{2} \partial_{xx} \Phi(\bar{\rho})$$

with Φ the mean local jump rate $\Phi(\rho) = \mathbb{E}_{\nu_\rho}[g(\eta(0))]$.

Rate of convergence?

- [Central limit fluctuations in non-equilibrium - Ferrari, Presutti, Vares; 1988]: Fluctuation density field

  $$Y_t^N = \frac{1}{\sqrt{N}} \sum_k \delta_{\frac{k}{N}}(x)[\eta_{tN^2}(k) - <\eta_{tN^2}(k)>]$$

  for $t \geq 0$, $< \cdot >$ the expectation. Then,

  $$\mathscr{L}(Y^N) \rightharpoonup^* \mathscr{L}(Y) \text{ for } N \to \infty$$

  with Y the solution to

  $$dY_t = \partial_{xx}(\Phi'(\bar{\rho}_t(x))Y_t)\,dt + \partial_x(\sqrt{\Phi(\bar{\rho}_t(x))}dW_t)$$

  with dW space-time white noise.

- Therefore, expect

  $$d(\mu_t^N, \bar{\rho}_t\,dx) \approx N^{-\frac{1}{2}}.$$

- [Large deviation principle, Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995]: Let now ρ_0 constant. Then, informally,

$$\mathbb{P}[\mu^N \approx \rho\, dx] \approx \exp\{-N\, I_0(\rho\, dx)\},$$

with rate function

$$I_0(\rho\, dx) = \inf\left\{ \int_0^{t_0}\int_{\mathbb{T}} |g|^2 dxds : g \in L^2_{t,x},\ \underbrace{\partial_t\rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)g)}_{\text{"skeleton equation"}} \right\}.$$

Link to stochastic PDE

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Informal justification:

1. Physics: Fluctuation-dissipation relation, "fluctuating hydrodynamics"
2. Mean behavior / law of large numbers

$$\rho^N \to \bar{\rho} \quad \text{as } N \to \infty.$$

3. Central limit fluctuations: $Y^N := \sqrt{N}(\rho^N - \bar{\rho})$. Then, $\mathscr{L}(Y^N) \rightharpoonup^* \mathscr{L}(Y)$ with

$$\partial_t Y = \partial_{xx}\left(\Phi'(\bar{\rho})Y\right) + \partial_x\left(\sqrt{\Phi(\bar{\rho})}dW_t\right).$$

4. Large deviations: See below, large deviations of $(\star)$ are the same as for μ^N.

- Consider

$$\partial_t \rho^N = \partial_{xx}\left(\Phi(\rho^N)\right) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho^N)}dW_t\right).$$

- Informally applying the contraction principle to the solution map

$$F : \frac{1}{\sqrt{N}}dW \mapsto \rho$$

yields as a rate function

$$I(\rho) = \inf\{I_{dW}(g) : F(g) = \rho\}.$$

- Schilder's theorem for Brownian sheet suggests

$$I_{dW}(g) = \int_0^T \int_{\mathbb{T}} |g|^2 \, dxdt.$$

- Get LDP with rate function

$$I(\rho) = \inf\left\{\int_0^T \int_{\mathbb{T}} |g|^2 \, dxdt : \partial_t \rho = \partial_{xx}(\Phi(\rho)) + \partial_x\left(\sqrt{\Phi(\rho)}g\right)\right\}.$$

- Obstacle
$$\partial_t\rho = \partial_{xx}(\Phi(\rho)) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho)}dW_t\right)$$
  1. not well-posed, supercritical -> no regularity structures
  2. Renormalization? Does renormalization appear in rate function? E.g. compare $\Phi^4_{2/3}$ [Hairer, Weber; 2014].

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- Ansatz: joint limit "small noise, ultraviolet cutoff"

$$\partial_t \rho^{N,K} = \partial_{xx}\left(\Phi(\rho^{N,K})\right) + \frac{1}{\sqrt{N}}\partial_x\left(\sqrt{\Phi(\rho^{N,K})}\circ dW^K_t\right)$$

where $W^K = \sum_{k=1}^K e_k \beta^k$ is a spectral (smooth) approximation of $W = \sum_{k=1}^\infty e_k \beta^k$.
- Gives the correct rate function for $\frac{1}{N} << \frac{1}{K}$.

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Note: This is a particular case in which the link between *Macroscopic fluctuation theory* [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015] and *fluctuating hydrodynamics* [Landau-Lifshitz 1973, Spohn 1991] can be made rigorous.

Two ways to the LDP

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- In the following concentrate on the case
$$\Phi(\rho) = \rho^m, \quad m \geq 1.$$
- We consider stochastic PDE of the type
$$\partial_t \rho^{N,K} = \Delta\left((\rho^{N,K})^m\right) + \frac{1}{\sqrt{N}} \operatorname{div}\left((\rho^{N,K})^{\frac{m}{2}} \circ dW_t^K\right), \qquad (*)$$
on $\mathbb{T}^d \times (0,\infty)$, where $W^K = \sum_{k=1}^K e_k \beta^k$.
- Pathwise well-posedness of $(*)$: [Lions, Souganidis; 1998ff], [Lions, Perthame, Souganidis; 2013], [Lions, Perthame, Souganidis; 2014], [G., Souganidis; 2014], [G., Souganidis; 2015], [G., Fehrman; 2017], [Dareiotis, G.; 2019].

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Two ways to the LDP:

1. Γ-convergence of the rate functional: $N \uparrow \infty$ yields LDP for $(*)$ with rate function

$$I^K(\rho) = \inf\left\{\int_0^T \int_{\mathbb{T}} |g|^2\, dxdt : \partial_t \rho = \partial_{xx}(\Phi(\rho)) + \partial_x\left(\sqrt{\Phi(\rho)} P^K g\right)\right\}.$$

Then consider $K \uparrow \infty$.

2. Joint scaling: Weak convergence approach to LDP ($\frac{1}{N} << \frac{1}{K}$).

- Both approaches crucially depend on understanding the skeleton PDE.
- The skeleton equation

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}\left(\rho^{\frac{m}{2}} g(t,x)\right) \qquad (*)$$
$$\rho(0,x) = \rho_0(x),$$

  with $g \in L^2_{t,x}$?
- This leads to the key problem

Problem

1. *Existence and uniqueness of solutions to* $(*)$.
2. *Stability of solutions: Let* $g^n \rightharpoonup g$ *in* $L^2_{t,x}$ *with corresponding solutions* ρ^n, ρ. *Then*

$$\rho^n \to \rho$$

   in $L^\infty_t L^1_x$.

- Difficulty: Stable a-priori bound? L^p framework does not work.
- Do we expect non-concentration of mass / well-posedness?

Scaling and criticality of the skeleton equation

- We consider
$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d$$
with $g \in L^q(\mathbb{R}_{+,t}; L^p(\mathbb{R}^d_x; \mathbb{R}^d_x))$ and $\rho_0 \in L^r(\mathbb{R}^d_x)$.
- Via rescaling ("zooming in"):
  - $p = q = 2$ is critical.
  - $r = 1$ is critical, $r > 1$ is supercritical.

Consider

$$d\rho^N = \Delta(\rho^N)^m\, dt + \frac{1}{\sqrt{N}}\mathrm{div}\left(\Phi^{\frac{1}{2}}_{n(N)}(\rho^N)\circ dW^{K(N)}(t)\right).$$

Theorem (Large deviation principle)

Let $K(N), n(N) \to \infty$ with $\frac{K(N)^3}{N} \to 0$ for $N \to \infty$. For $\rho_0 \in L^{m+1}(\mathbb{T}^d)$ and $\rho \in L^\infty([0,T]; L^1(\mathbb{T}^d))$ let

$$I_{\rho_0}(\rho) := \inf\left\{\frac{1}{2}\int_0^T \|g(s)\|^2_{L^2_x} ds :\ g \in L^2_{t,x}, \partial_t\rho = \Delta\rho^m + div(\rho^{\frac{m}{2}} g)\right\}.$$

Then,

1. *For all $\rho_0 \in L^{m+1}(\mathbb{T}^d)$, $\rho \mapsto I_{\rho_0}(\rho)$ is a good rate function on $L^\infty([0,T]; L^1(\mathbb{T}^d))$.*
2. *The family $\{\rho^N\}$ satisfies the large deviation principle on $L^\infty([0,T]; L^1(\mathbb{T}^d))$ with rate function I_{ρ_0}, uniformly on compact subsets of $L^{m+1}(\mathbb{T}^d)$.*

K. Dareiotis and B. Gess.
Nonlinear diffusion equations with nonlinear gradient noise.
arXiv:1811.08356 [math], Nov. 2018.

B. Fehrman and B. Gess.
Well-posedness of nonlinear diffusion equations with nonlinear, conservative noise.
Archive for Rational Mechanics and Analysis, 233(1):249–322, 2019.

B. Fehrman and B. Gess.
Large deviations for conservative stochastic PDE and non-equilibrium fluctuations.
arXiv:1910.11860 [math], Mar. 2020.

B. Gess and P. E. Souganidis.
Scalar conservation laws with multiple rough fluxes.
Commun. Math. Sci., 13(6):1569–1597, 2015.

B. Gess and P. E. Souganidis.
Stochastic non-isotropic degenerate parabolic–hyperbolic equations.
Stochastic Process. Appl., 127(9):2961–3004, 2017.