

Nonlinear stochastic partial differential equations in nonequilibrium statistical mechanics

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joint work with: Ben Fehrman.
[G., Fehrman; arxiv, 2019].

Introduction: Large deviations in zero range process

- 1 Introduction: Large deviations for the zero range process
 - Fluctuations in the zero range process
 - Link to stochastic PDE
- 2 Two ways to the LDP
 - Scaling and criticality for the skeleton equation
 - Well-posedness of the skeleton equation

The zero range process

(could also consider simple exclusion, independent particles).

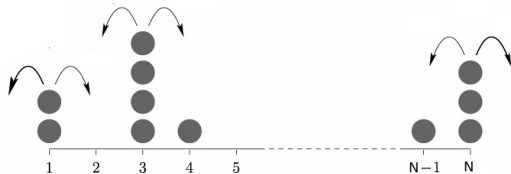


Figure: Harris, Rákos, Schütz; 2005

- Discrete d -dim. torus $\mathbb{T}_N^d := (\mathbb{Z}/(N\mathbb{Z}))^d$
- State space $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N^d}$, i.e. configurations $\eta : \mathbb{T}_N^d \rightarrow \mathbb{N}_0$: System in state η if container x contains $\eta(x)$ particles.
- Local jump rate function $g : \mathbb{N}_0 \rightarrow \mathbb{R}_0^+$, with $g(k) = 0$ iff $k = 0$, g Lipschitz continuous.

- Translation invariant, asymmetric, zero mean transition probability

$$p(x, y) = p(x - y), \quad \sum_k kp(k) = 0.$$

- Zero range process with jump rate g is the Markov jump process η_t on $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N^d}$ with generator

$$L_N f(\eta) = \sum_{x, y \in \mathbb{T}_N^d} (f(\eta^{x, y}) - f(\eta)) g(\eta(x)) p(x, y),$$

where $f : \mathbb{M}_N \rightarrow \mathbb{R}$, $\eta^{x, y} = \eta - 1_{\{x\}} + 1_{\{y\}}$.

- Invariant probability measures ν_ρ (product structure and explicit), indexed by density $\rho \geq 0$.

- Hydrodynamic limit?

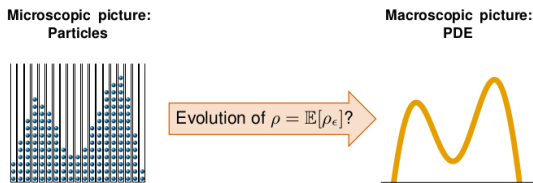


Figure: see Zimmer et. al.

- Empirical density field

$$\mu_t^N(x) := \frac{1}{N^d} \sum_k \delta_{\frac{k}{N}}(x) \eta_{tN^2}(k).$$

- Assume that $\mathcal{L}(\eta_0^N)$ is associated to a smooth profile ρ_0 with $0 < \rho_- \leq \rho_0 \leq \rho_+ < \infty$, that is, $\mathcal{L}(\eta_0^N)$ around Nx converges to $\nu_{\rho_0(x)}$ in probability.

Theorem (Hydrodynamic limit - Ferrari, Presutti, Vares; 1987)

For $\delta > 0$, G continuous, bounded,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left[\left| \langle \mu_t^N, G \rangle - \langle \rho_t dx, G \rangle \right| > \delta \right] = 0$$

where $\rho(t, u)$ is the unique weak solution to

$$\begin{aligned} \partial_t \rho &= \frac{1}{2} \partial_{xx} \Phi(\rho) \\ \rho(0) &= \rho_0 \end{aligned}$$

with Φ the mean local jump rate

$$\Phi(\rho) = \mathbb{E}_{\nu_\rho} [g(\eta(0))].$$

E.g. $\Phi(\rho) = \rho |\rho|^{m-1}$.

- E.g. for constant intensity ($g(k) = 1_{k>0}$) get $\Phi(\rho) = \frac{\rho}{1+\rho}$.
- Independent particles, simple exclusion process: $\Phi(\rho) = \rho$.

Theorem (Central limit fluctuations in non-equilibrium - Ferrari, Presutti, Vares; 1988)

Let η be the zero range process with constant intensity with η_0^N in local equilibrium with smooth profile ρ_0 . Define the fluctuation density field

$$Y_t^N = \frac{1}{\sqrt{N}} \sum_k \delta_{\frac{k}{N}}(x) [\eta_{tN^2}(k) - \langle \eta_{tN^2}(k) \rangle]$$

for $t \geq 0$, $\langle \cdot \rangle$ the expectation. Let \mathcal{P}^N the law of Y^N on $D([0, \infty), \mathcal{S}'(\mathbb{T}))$. Then,

$$\mathcal{P}^N \xrightarrow{*} \mathcal{P} \text{ for } N \rightarrow \infty$$

with \mathcal{P} a Gaussian, supported on $C([0, \infty), \mathcal{S}'(\mathbb{T}))$, the martingale solution to

$$dY_t = \frac{1}{2} \partial_{xx}(\Phi'(\rho_t(x)) Y_t) dt + \partial_x(\sqrt{\Phi(\rho_t(x))} dW_t)$$

with initial condition Y_0 is Gaussian, centered and covariance

$$\mathcal{X}(\rho_0(r)) = \mathbb{E}_{\nu_{\rho_0(r)}}(\eta_0(\eta_0 - \rho_0(r))).$$

- Let now ρ_0 constant.
- For $\mu \in D([0, t_0]; \mathcal{M}_+)$, $\rho_t = d\mu_t/dx$ let

$$I_0(\mu) = \inf \left\{ \int_0^{t_0} \int |g|^2 dx ds : g \in L_{t,x}^2, \underbrace{\partial_t \rho = \partial_{xx} \Phi(\rho) + \partial_x (\Phi^{\frac{1}{2}}(\rho) g)}_{\text{"skeleton equation"}} \right\}.$$

- E.g. independent particles: $\Phi(\rho) = \rho$.

Theorem (Large deviation principle, Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995)

For every measurable $A \subseteq D([0, t_0], \mathcal{M}_+)$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}[\mu^N \in \bar{A}] \leq - \inf_{\mu \in \bar{A}} I_0(\mu) \leq - \inf_{\mu \in A^\circ} I_0(\mu) \leq \liminf_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{P}[\mu^N \in A^\circ].$$

Informally, take $A = \{\rho dx\}$,

$$\mathbb{P}[\mu^N \approx \rho dx] \approx \exp\{-N I_0(\rho dx)\}.$$

Link to stochastic PDE

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Link to stochastic PDE

- Large deviations [Dirr, Stamatakis, Zimmer; 2016] as special case of MFT [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015],

$$\partial_t \rho = \partial_{xx}(\Phi(\rho)) + \frac{1}{\sqrt{N}} \partial_x \left(\sqrt{\Phi(\rho)} dW_t \right).$$

- Model case: Dean-Kawasaki, independent particles, $\Phi(\rho) = \rho$, i.e.

$$\partial_t \rho = \partial_{xx} \rho + \frac{1}{\sqrt{N}} \partial_x (\sqrt{\rho} dW_t).$$

Informally same rate function as the zero range process:

- Informally applying the contraction principle to the solution map

$$F : \frac{1}{\sqrt{N}} dW \mapsto \rho$$

yields as a rate function

$$I(\rho) = \inf \{ I_{dW}(g) : F(g) = \rho \}.$$

- Schilder's theorem for Brownian sheet suggests

$$I_{dW}(g) = \int_0^T \int_{\mathbb{T}} |g|^2 dx dt.$$

- Get

$$I(\rho) = \inf \left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dx dt : \partial_t \rho = \partial_{xx} (\Phi(\rho)) + \partial_x (\sqrt{\Phi(\rho)} g) \right\}.$$

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- Obstacle

$$\partial_t \rho = \partial_{xx} (\Phi(\rho)) + \frac{1}{\sqrt{N}} \partial_x (\sqrt{\Phi(\rho)} dW_t)$$

- 1 not well-posed, supercritical -> no regularity structures
- 2 Renormalization? Does renormalization appear in rate function? E.g. compare $\Phi_{2/3}^4$ [Hairer, Weber; 2014].

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- Ansatz: joint limit “small noise, ultraviolet cutoff”

$$\partial_t \rho^{N,K} = \partial_{xx} (\Phi(\rho^{N,K})) + \frac{1}{\sqrt{N}} \partial_x (\sqrt{\Phi(\rho^{N,K})} \circ dW_t^K),$$

where $W^K = \sum_{k=1}^K e_k \beta^k$ is a spectral (smooth) approximation of $W = \sum_{k=1}^{\infty} e_k \beta^k$.

- Gives the correct rate function for $\frac{1}{N} \ll \frac{1}{K}$.

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Note: This is a particular case in which the link between *Macroscopic fluctuation theory* [Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; 2015] and *fluctuating hydrodynamics* [Landau-Lifshitz 1973, Spohn 1991] can be made rigorous.

Two ways to the LDP

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- In the following concentrate on the case

$$\Phi(\rho) = \rho^m, \quad m \geq 1.$$

- We consider stochastic PDE of the type

$$\partial_t \rho^{N,K} = \Delta \left((\rho^{N,K})^m \right) + \frac{1}{\sqrt{N}} \operatorname{div} \left((\rho^{N,K})^{\frac{m}{2}} \circ dW_t^K \right), \quad (*)$$

on $\mathbb{T}^d \times (0, \infty)$, where $W^K = \sum_{k=1}^K e_k \beta^k$.

- Pathwise well-posedness of (*): [Lions, Perthame, Souganidis; 2013], [Lions, Perthame, Souganidis; 2014], [G., Souganidis; 2014], [G., Souganidis; 2015], [G., Fehrman; 2017], [Dareiotis, Gess; 2019].

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Two ways to the LDP:

- 1 Γ -convergence of the rate functional: $N \uparrow \infty$ yields LDP for (*) with rate function

$$I^K(\rho) = \inf \left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dx dt : \partial_t \rho = \partial_{xx} (\Phi(\rho)) + \partial_x \left(\sqrt{\Phi(\rho)} P^K g \right) \right\}.$$

Then consider $K \uparrow \infty$.

- 2 Joint scaling: Weak convergence approach to LDP ($\frac{1}{N} \ll \frac{1}{K}$).

- Both approaches crucially depend on understanding the skeleton PDE.
- The skeleton equation

$$d\rho = \Delta \rho^m dt + \operatorname{div} \left(\rho^{\frac{m}{2}} g(t, x) \right) dt \quad (*)$$

$$\rho(0, x) = \rho_0(x),$$

with $g \in L^2_{t,x}$?

- This leads to the key problem

Problem

- 1 Existence and uniqueness of solutions to (*).
- 2 Stability of solutions: Let $g^n \rightarrow g$ in $L^2_{t,x}$ with corresponding solutions ρ^n, ρ . Then

$$\rho^n \rightarrow \rho$$

in $L^\infty_t L^1_x$.

- Difficulty: Stable a-priori bound? L^p framework does not work.
- Do we expect non-concentration of mass / well-posedness?

Scaling and criticality of the skeleton equation

- We consider

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d$$

with $g \in L^q(\mathbb{R}_{+,t}; L^p(\mathbb{R}_x^d; \mathbb{R}_x^d))$ and $\rho_0 \in L^r(\mathbb{R}_x^d)$.

- Via rescaling (“zooming in”):
 - $p = q = 2$ is critical.
 - $r = 1$ is critical, $r > 1$ is supercritical.

Apriori-bounds and energy space

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$$\partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d \quad (*)$$

with $g \in L^2(\mathbb{R}_{+,t}; L^2(\mathbb{R}_x^d; \mathbb{R}_x^d))$.

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- Use entropy-entropy dissipation: Evolution of entropy given by $\int_{\mathbb{T}^d} \log(\rho) \rho$. Informally gives

$$\int_{\mathbb{T}^d} \log(\rho) \rho dx \Big|_0^t + \int_0^t \int_{\mathbb{T}^d} (\nabla \rho^{\frac{m}{2}})^2 \lesssim \int_0^t \int_{\mathbb{T}^d} g^2.$$

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- Caution: Can only be true for non-negative solutions.
- Non-standard weak solutions, rewriting (*) as

$$\partial_t \rho = 2 \operatorname{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d$$

- Conclusion: Have to prove uniqueness within this class of solutions.

Ansatz for uniqueness: Show that every weak solution is a renormalized entropy solution (extending the concepts of DiPerna-Lions, Ambrosio to nonlinear PDE).

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Let

$$\chi(t, x, \xi) = 1_{0 < \xi < \rho(x, t)} - 1_{\rho(x, t) < \xi < 0}.$$

Then, informally,

$$\partial_t \chi = m \xi^{m-1} \Delta_x \chi - g(x, t) (\partial_\xi \xi^{\frac{m}{2}}) \nabla_x \chi + (\nabla_x g(x, t)) \xi^{\frac{m}{2}} \partial_\xi \chi + \partial_\xi p$$

with p parabolic defect measure

$$p = \delta(\xi - \rho) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2.$$

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with p parabolic defect measure

$$p = \delta(\xi - \rho) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2.$$

Or on “fluid level”, informally, for F convex,

$$\partial_t F(\rho) - \Delta_x F^1(\rho) + g(x, t) \nabla_x F^2(\rho) + (\nabla_x g)(x, t) F^3(\rho) = - \int_\xi F''(\xi) p \leq 0$$

where $(F^m)' = m f(\xi) \xi^{m-1}$, $(F^2)' = f(\xi) (\partial_\xi \xi^{\frac{m}{2}})$, $F^3 = f(\xi) \xi^{\frac{m}{2}}$.

- How to make that rigorous? Take convolution

$$\rho^\varepsilon = \varphi^\varepsilon *_x \rho.$$

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- Commutator errors,

$$\begin{aligned} \partial_t \rho^\varepsilon &= \varphi^\varepsilon * \partial_t \rho = \varphi^\varepsilon * (\Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g)) \\ &= \Delta(\varphi^\varepsilon * \rho^m) + \operatorname{div}(\varphi^\varepsilon * (\rho^{\frac{m}{2}} g)) \\ &= \Delta(\rho^\varepsilon)^m + \operatorname{div}((\rho^\varepsilon)^{\frac{m}{2}} g) \\ &\quad + \Delta(\varphi^\varepsilon * \rho^m) - \Delta(\rho^\varepsilon)^m \\ &\quad + \operatorname{div}((\varphi^\varepsilon * \rho^{\frac{m}{2}}) g) - \operatorname{div}((\rho^\varepsilon)^{\frac{m}{2}} g) \\ &\quad + \operatorname{div}(\varphi^\varepsilon * (\rho^{\frac{m}{2}} g)) - \operatorname{div}((\varphi^\varepsilon * \rho^{\frac{m}{2}}) g). \end{aligned}$$

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- Note: Additional commutator errors by commuting convolution and nonlinearities!
- Commutator estimate using non-standard (optimal) regularity $\rho^{\frac{m}{2}} \in L_t^2 \dot{H}_x^1$
- Additional renormalization step to compensate low time integrability $\rho^{\frac{m}{2}} g \in L_t^1 L_x^1$.

Uniqueness for renormalized entropy solutions (variable doubling)

- Additional errors from space-inhomogeneity (with little regularity)
- Note: Entropy dissipation measure

$$q(x, \xi, t) = \delta(\xi - \rho(x, t)) 4 \frac{\xi^m}{\xi^{m-1}} |\nabla \rho^{\frac{m}{2}}|^2$$

does not satisfy

$$\lim_{|\xi| \rightarrow \infty} \int_{t,x} q(x, \xi, t) dx dt = 0.$$

- Established arguments [Chen, Perthame; 2003] not applicable.

Theorem (The skeleton equation)

Let $g \in L^2([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$, $\rho_0 \in L^1(\mathbb{T}^d)$ non-negative and $\int \rho_0 \log(\rho_0) dx < \infty$, $m \in [1, \infty)$.

1 There is a unique weak solution

$$\partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d \quad (*)$$

For two weak solutions $\rho^1, \rho^2 \in L^\infty([0, T]; L^1(\mathbb{T}^d))$ we have

$$\|\rho^1 - \rho^2\|_{L^\infty([0, T]; L^1(\mathbb{T}^d))} \leq \|\rho_0^1 - \rho_0^2\|_{L^1(\mathbb{T}^d)}.$$

2 Let $\{g_n\}_{n \in \mathbb{N}} \subseteq L^2([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$ with

$$\lim_{n \rightarrow \infty} g_n = g \text{ weakly in } L^2([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$$

and let $\rho_n \in L^1([0, T]; L^1(\mathbb{T}^d))$ be the corresponding solutions with control g_n . Then,

$$\lim_{n \rightarrow \infty} \rho_n = \rho \text{ strongly in } L^1([0, T]; L^1(\mathbb{T}^d))$$

where $\rho \in L^1([0, T]; L^1(\mathbb{T}^d))$ is the solution with control g .

Consider

$$d\rho^N = \Delta(\rho^N)^m dt + \frac{1}{\sqrt{N}} \operatorname{div} \left(\Phi_{n(N)}^{\frac{1}{2}}(\rho^N) \circ dW^{K(N)}(t) \right).$$

Theorem (Large deviation principle)

Let $K(N), n(N) \rightarrow \infty$ with $\frac{K(N)^3}{N} \rightarrow 0$ for $N \rightarrow \infty$. For $\rho_0 \in L^{m+1}(\mathbb{T}^d)$ and $\rho \in L^\infty([0, T]; L^1(\mathbb{T}^d))$ let

$$I_{\rho_0}(\rho) := \inf \left\{ \frac{1}{2} \int_0^T \|g(s)\|_{L_x^2}^2 ds : g \in L_{t,x}^2, \partial_t \rho = \Delta \rho^m + \operatorname{div}(\rho^{\frac{m}{2}} g) \right\}.$$

Then,

- 1 For all $\rho_0 \in L^{m+1}(\mathbb{T}^d)$, $\rho \mapsto I_{\rho_0}(\rho)$ is a good rate function on $L^\infty([0, T]; L^1(\mathbb{T}^d))$.
- 2 The family $\{\rho^N\}$ satisfies the large deviation principle on $L^\infty([0, T]; L^1(\mathbb{T}^d))$ with rate function I_{ρ_0} , uniformly on compact subsets of $L^{m+1}(\mathbb{T}^d)$.



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