Non-equilibrium large deviations and PDEs with irregular coefficients

Benjamin Gess
Universität Bielefeld & Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig

AIMS conference, Wilmington, June 2023

joint work with Ben Fehrman [https://arxiv.org/abs/1910.11860]
and Daniel Heydecker [https://arxiv.org/abs/2303.11289]

slides online: bgess.de → talks

Max-Planck-Institut für Mathematik in den Naturwissenschaften

Universität Bielefeld

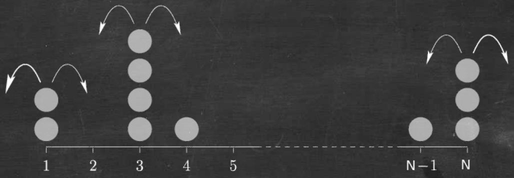
Content

From large deviations to PDEs with irregular drift

PDEs with irregular drift

From large deviations to PDEs with irregular drift

The zero range process (could also consider simple exclusion, independent particles, ..).



- State space $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N}$, i.e. configurations $\eta : \mathbb{T}_N \to \mathbb{N}_0$: System in state η if container k contains $\eta(k)$ particles.
- Local jump rate function $g : \mathbb{N}_0 \to \mathbb{R}_0^+$.
- Translation invariant, asymmetric, zero mean transition probability

$$p(k,l) = p(k-l), \quad \sum_k kp(k) = 0.$$

- Markov jump process $\eta(t)$ on $\mathbb{M}_N$.
- $\eta(k,t) = \text{number of particles in box } k \text{ at time } t$.

- **Hydrodynamic limit?** Multi-scale dynamics

- Empirical density field: $\mu^N(x,t) := \frac{1}{N}\sum_k \delta_{\frac{k}{N}}(x)\eta(k,tN^2)$.
- [Hydrodynamic limit - Ferrari, Presutti, Vares; 1987]

$$\mu^N(t) \rightharpoonup^* \bar{\rho}(t)\,dx$$

  with

$$\partial_t \bar{\rho} = \partial_{xx}\Phi(\bar{\rho})$$

  with Φ the mean local jump rate $\Phi(\rho) = \mathbb{E}_{\nu_\rho}[g(\eta(0))]$.
- Fluctuations
  - Rare events / large deviations?
  - Central limit theorem.

Rare events: (Im-)probability to observe a fluctuation ρ:

$$\mathbb{P}[\mu^N \approx \rho] = e^{-N\, I(\rho)} \quad N \text{ large}$$

A bit more precisely, for every open set O,

$$\mathbb{P}[\mu^N \in \bar{O}] \lesssim e^{-N \inf_{\rho \in \bar{O}} I(\rho)}$$

$$e^{-N \inf_{\rho \in O} I(\rho)} \lesssim \mathbb{P}[\mu^N \in O]$$

Zero range process

$$I(\rho) = \inf\{ \underbrace{\int_{t,x} |\partial_x H|^2 \Phi(\rho)}_{=:\|H\|_{H^1_{\Phi(\rho)}}} : \quad \underbrace{\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x H)}_{\text{"controlled nonlinear Fokker-Planck equation"}} \quad \}.$$

e.g.

$$I(\rho) = \inf\{ \int_{t,x} |\partial_x H|^2 \rho \, : \partial_t \rho = \partial_{xx}\rho + \partial_x(\rho \partial_x H)\}.$$

Theorem ([Large deviation principle, Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995])

For every open set $O \subseteq D([0,T], \mathcal{M}_+)$ we have

$$\mathbb{P}[\mu^N \in \bar{O}] \lesssim e^{-N \inf_{\rho \in \bar{O}} I(\rho)}$$

$$\mathbb{P}[\mu^N \in \bar{O}] \lesssim e^{-N \inf_{\rho \in \bar{O}} I(\rho)}$$

$$e^{-N \inf_{\rho \in O} J(\rho)} \lesssim \mathbb{P}[\mu^N \in O]$$

where $J = \overline{I_{|A}}$ and A is the set of nice fluctuations $\mu = \rho\, dx$ with ρ a solution to

$$\partial_t \rho = \partial_{xx} \Phi(\rho) + \partial_x(\Phi(\rho)\partial_x H)$$

for some $\underline{H \in C^{1,3}_{t,x}}$.

This is a frequently observed problem: E.g. Fluctuations around Boltzmann equation [Rezakhanlou 1998], [Bodineau, Gallagher, Saint-Raymond, Simonella 2020].
Counter-examples for Boltzmann [Heydecker; 2021].

Problem:

$$I = J = \overline{I_{|A}}\,?$$

Existence of a "recovery sequence"? Given fluctuation ρ so that $I(\rho dx) < \infty$, i.e. for some $H \in H^1_{\Phi(\rho)}$,

$$\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x \underbrace{H}_{\in H^1_{\Phi(\rho)}}).$$

Need to find sequence of nice fluctuations $\rho^\varepsilon \in A$ so that $\rho^\varepsilon \to \rho$ and $I(\rho^\varepsilon) \to I(\rho)$. That is, find $H^\varepsilon \in C^{1,3}([0,T] \times \mathbb{T})$ so that

$$I(\rho^\varepsilon) = \|H^\varepsilon\|_{L^2_t H^1_{\Phi(\rho)}} \to \|H\|_{L^2_t H^1_{\Phi(\rho)}} = I(\rho)$$

and

$$\partial_t \rho^\varepsilon = \partial_{xx}\Phi(\rho^\varepsilon) + \partial_x(\Phi(\rho^\varepsilon)\partial_x H^\varepsilon)$$

satisfies $\rho^\varepsilon \to \rho$.

Difficult: Open problem for the zero range process since [Benois, Kipnis, Landim; 1995].

Conclusion: Stability and uniqueness of PDEs with irregular coefficients

$$\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x \underbrace{H}_{\in H^1_{\Phi(\rho)}}).$$

Rewrite the rate function:

$$I(\rho) = \inf\{\int_{t,x} |\partial_x H|^2 \Phi(\rho) \ : \partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x H)\}$$

$$= \inf\{\int_{t,x} |\underbrace{\Phi^{\frac{1}{2}}(\rho)\partial_x H}_{=:g}|^2 \ : \partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)\underbrace{\Phi^{\frac{1}{2}}(\rho)\partial_x H}_{g})\}$$

$$= \inf\{\int_{t,x} |g|^2 \ : \underbrace{\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)g)}_{\text{"skeleton equation"}}\}$$

e.g.

$$I(\rho) = \inf\{\int_{t,x} |g|^2 \ : \partial_t \rho = \partial_{xx}\rho + \partial_x(\rho^{\frac{1}{2}} g)\}$$

Stability: $g \mapsto \rho$, $L^2_{t,x} \to L^1_{t,x}$ continuous? I.e. Stability and uniqueness of a PDE with irregular coefficients $g \in L^2_{t,x}$.

Observations:

1. Representation via skeleton equation is natural coming from conservative SPDEs.
2. Skeleton equation is always nonlinear and singular.
3. Stability properties are better studied via the skeleton PDE

PDEs with irregular drift

Skeleton equation

$$\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho) \underbrace{g}_{\in L^2_{t,x}}).$$

How difficult is the well-posedness?

- Difficulty: Stable a-priori bound? L^p framework does not work.
- Do we expect non-concentration of mass / well-posedness?

Scaling and criticality of the skeleton equation

- We consider, $\Phi(\rho) = \rho^m$,

$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g)$$

  with $g \in L^q_t L^p_x$ and $\rho_0 \in L^r_x$.
- Via rescaling ("zooming in"):
  - $p = q = 2$ is critical.
  - $r = 1$ is critical, $r > 1$ is supercritical.

Recall: [Le Bris, Lions; CPDE 2008], [Karlssen, Risebro, Ohlberger, Chen, ...]

$$\partial_t \rho = \frac{1}{2} D^2 : (\sigma\sigma^* \rho) + \mathrm{div}(\rho g)$$

needs $g \in W^{1,1}_{loc,x}$, $\mathrm{div}\, g \in L^\infty$.

Overview of ingredients of the proof:

- **Part 1:** Apriori-bounds; entropy-entropy dissipation estimates
- **Part 2:** Extending the concepts of DiPerna-Lions, Ambrosio, Le Bris-Lions to nonlinear PDE (but going beyond).
- **Part 3:** Uniqueness for renormalized entropy solutions (variable doubling): New treatment of kinetic dissipation measure. Exploit finite *singular* moments.

Part 2: Renormalization

Recall: Linear case [DiPerna, Lions, Invent. 1989; Ambrosio Invent. 2004]

$$\partial_t \rho = \mathrm{div}(\rho g).$$

Then ρ is a renormalized solution, if for all smooth f we have

$$\partial_t f(\rho) = \mathrm{div}(f(\rho) g) - (f(\rho) + f'(\rho)\rho)\mathrm{div} g.$$

Let ρ be a weak solution to

$$\partial_t \rho = 2\mathrm{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \mathrm{div}(\rho^{\frac{m}{2}} g).$$

Show that every weak solution is a renormalized (= kinetic) solution (merging renormalization [DiPerna, Lions; Ambrosio] with kinetic solutions [Lions, Perthame, Tadmor, J. Amer. Math. Soc. 1994]).

Let

$$\chi(t,x,\xi) = f_\xi(\rho(x,t)) = 1_{0<\xi<\rho(x,t)} - 1_{\rho(x,t)<\xi<0}.$$

Then, informally,

$$\partial_t \chi = m\xi^{m-1}\Delta_x \chi - g(x,t)(\partial_\xi \xi^{\frac{m}{2}})\nabla_x \chi + (\nabla_x g(x,t))\xi^{\frac{m}{2}}\partial_\xi \chi + \partial_\xi q$$

with p parabolic defect measure

$$q = \delta(\xi - \rho(x,t))|\nabla \rho^{\frac{m+1}{2}}|^2.$$

- Note: Additional commutator errors by commuting convolution and nonlinearities.
- Commutator estimate using non-standard (optimal) regularity $\rho^{\frac{m}{2}} \in L^2_t \dot{H}^1_x$
- Additional renormalization step to compensate low time integrability $\rho^{\frac{m}{2}} g \in L^1_t L^1_x$.

Theorem

A function $\rho \in L^\infty_t L^1_x$ is a weak solution to

$$\partial_t \rho = 2div(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + div(\rho^{\frac{m}{2}} g)$$

if and only if ρ is a renormalized entropy solution (kinetic solution).

Theorem (The skeleton equation, Fehrman, G. 2022)

Let $g \in L^2_{t,x}$, ρ_0 non-negative and $\int \rho_0 \log(\rho_0) dx < \infty$. There is a unique weak solution to

$$\partial_t \rho = \Delta \Phi(\rho) + div(\Phi^{\frac{1}{2}}(\rho) g).$$

The map $g \mapsto \rho$, $L^2_{t,x} \to L^1_{t,x}$, is weak-strong continuous. E.g. including all $\Phi(\rho) = \rho^m$, $m \in [1, \infty)$.

Theorem (LDP for zero range process, G., Heydecker, 2023)

The rescaled zero range process satisfies the full large deviations principle with rate function

$$I(\rho) = \|\partial_t \rho - \partial_{xx} \Phi(\rho)\|^2_{H^{-1}_{\Phi(\rho)}}.$$

References

B. Fehrman and B. Gess.
Non-equilibrium large deviations and parabolic-hyperbolic PDE with irregular drift.
arXiv:1910.11860 [math], Mar. 2022.

B. Gess and D. Heydecker.
A Rescaled Zero-Range Process for the Porous Medium Equation: Hydrodynamic Limit, Large Deviations and Gradient Flow, Mar. 2023.