Gradient flow structures and large deviations for porous media equations

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joint with Daniel Heydecker [arxiv] and Ben Fehrman [Invent. Math. 2023].

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Stochastic porous medium equation

Stochastic porous medium equation, $\alpha \geq 1$,

$$\partial_t \rho = \Delta \rho^\alpha + \underbrace{\text{noise}}_{?}.$$

Rewrite the PME as a gradient flow

$$\partial_t \rho = \Delta \rho^\alpha =_? -\nabla_{\mathcal{M}} \mathcal{H}(\rho) = -M(\rho) \frac{D\mathcal{H}}{D\rho}(\rho),$$

where $M(\rho)$ the inverse Riemannian tensor, $\mathcal{H}$ some entropy. Choose noise so that $\mu(d\rho) = \frac{1}{Z} e^{-\mathcal{H}(\rho)} d\rho$ becomes an invariant measure, i.e.

$$\partial_t \rho = -M(\rho) \frac{D\mathcal{H}}{D\rho}(\rho) + M^{\frac{1}{2}}(\rho) \diamond \xi.$$

Different gradient flow structures lead to different SPDEs.

Gradient flows for PME:

Brezis ['71]: $\mathcal{M} = H^{-1}$, $M(\rho) = -\Delta$, $\mathcal{H}(\rho) = \int \rho^{\alpha+1}$,

$$\partial_t \rho = \nabla \cdot (\nabla \rho^\alpha).$$

Otto ['01]: $\mathcal{M} = \mathcal{P}(\mathbb{T}^d)$, $M(\rho) = -\nabla \cdot (\rho \nabla \cdot)$, $\mathcal{H}(\rho) = \int \rho^\alpha$ pressure,

$$\partial_t \rho = \nabla \cdot (\rho \nabla \rho^{\alpha-1}).$$

"Thermodynamic metric": $\mathcal{M} = \mathcal{P}(\mathbb{T}^d)$, $M(\rho) = -\nabla \cdot (\rho^\alpha \nabla \cdot)$, $\mathcal{H}(\rho) = \mathcal{H}(\rho)$ Boltzmann entropy,

$$\partial_t \rho = \nabla \cdot (\rho^\alpha \nabla \log(\rho)).$$

Sideremark: Leads to fluctuating hydrodynamics SPDE

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\alpha/2} \diamond \xi).$$

"Thermodynamic metric"? Following [Otto; 2001], but replacing $\rho \mapsto \rho^\alpha$, does not lead to a Riemannian structure. Alternative

$$\partial_t \rho = -\nabla_{\mathcal{M}} \mathcal{H}(\rho). \qquad (\star)$$

Then

$$\partial_t \mathcal{H}(\rho) = -(\partial_t \rho, \frac{D\mathcal{H}}{D\rho})_{M(\rho)} \geq -|\partial_t \rho|_{M(\rho)} |\frac{D\mathcal{H}}{D\rho}|_{M(\rho)} \geq -\frac{1}{2}|\frac{D\mathcal{H}}{D\rho}|^2_{M(\rho)} - \frac{1}{2}|\partial_t \rho|^2_{M(\rho)}$$

with equality iff ρ solves $(\star)$.

Consequence: ρ is a gradient flow for $(\star)$ iff

$$\rho = \mathrm{argmin}_\rho \mathcal{I}(\rho) = \mathrm{argmin}_\rho \Big(\mathcal{H}(\rho_T) - \mathcal{H}(\rho_0) + \frac{1}{2}\int_0^T |\frac{D\mathcal{H}}{D\rho}|^2_{M(\rho)} + \frac{1}{2}\mathcal{A}(\rho)\Big)$$

where $\mathcal{A}(\rho) = \int_0^T |\dot\rho|^2_{M(\rho)} = \inf\{\|g\|^2_{L^2_{t,x}} : \partial_t \rho + M^{\frac{1}{2}}(\rho) g = 0\}$.

Definition: ρ is a "thermodynamic" gradient flow of

$$\partial_t \rho = \nabla \cdot (\rho^\alpha \nabla \log(\rho)).$$

iff

$$\rho = \mathrm{argmin}_\rho \mathcal{I}(\rho) = \mathrm{argmin}_\rho \Big(\mathcal{H}(\rho_T) - \mathcal{H}(\rho_0) + \frac{1}{2}\int_0^T \int \frac{|\nabla \rho^{\frac{\alpha+1}{2}}|^2}{\rho} dx dt + \frac{1}{2}\mathcal{A}(\rho)\Big).$$

Gradient flows and large deviations

Let μ^N be a stochastic particle system, in the mean converging to the solution to

$$\partial_t \bar{\rho} = \Delta \bar{\rho}^\alpha. \qquad (\star)$$

Rare events are the (im-)probability to observe a fluctuation ρ:

$$\mathbb{P}[\mu^N \approx \rho] = e^{-N\mathcal{I}(\rho)} \quad N \text{ large}$$

This characterizes the solution $\bar{\rho}$ to $(\star)$ as

$$\bar{\rho} = \operatorname{argmin}_\rho \mathcal{I}(\rho).$$

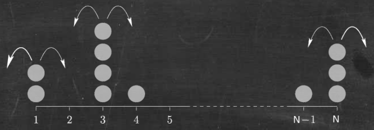
We say that a gradient flow structure corresponding to an energy $\mathcal{I}$ is thermodynamic, if there is a particle system μ^N satisfying an LDP with rate function $\mathcal{I}$.

Q: it is not (rigorously) known which, if any, of the gradient flow structures of PME are thermodynamic.

The porous medium equation as a hydrodynamic limit

Can we obtain the PME as a limit of a (stochastic) particle system?
E.g. [Suzuki, Ushiyama; 1993], [Ekhaus, Seppäläinen; 1996], [Oelschläger; 1990], [Gonçalves, Landim, Toninelli; 2009], [Gonçalves, Nahum, Simon; 2023].

The zero range process:



Local jump rate function $g(\eta) = \eta^\alpha : \mathbb{N}_0 \to \mathbb{R}_0^+$, $\alpha > 1$.

Translation invariant, asymmetric, zero mean transition probability $p(x, y)$.
Generator

$$L_N F(\eta) := \sum_{x,y \in \mathbb{T}_N^d} p(x, y)\eta^\alpha(x)(F(\eta^{x,y}) - F(\eta)).$$

Hydrodynamic limit Empirical density field:

$$\mu^N(x,t) := \Big(\frac{1}{N}\sum_k \delta_k(x)\eta(k,t)\Big)(xN, tN^2).$$

[Hydrodynamic limit - Ferrari, Presutti, Vares; 1987]

$$\mu^N(t) \rightharpoonup^* \bar\rho(t)\, dx$$

with

$$\partial_t \bar\rho = \Delta\Phi(\bar\rho)$$

with Φ the mean local jump rate $\Phi(\bar\rho) = \mathbb{E}_{\nu_{\bar\rho}}[\eta^\alpha(0)]$.

The Φ is non-degenerate: $\Phi' \geq c > 0$. Even if $g(\eta) = \eta^\alpha$ we do not see the porous medium equation, that is, $\Phi(\bar\rho) \neq \bar\rho^\alpha$, $\alpha \geq 1$.

The porous medium equation, $\alpha \geq 1$,

$$\partial_t \bar{\rho} = \Delta \bar{\rho}^\alpha$$

Scaling invariance of PME: Let $\tilde{\rho}(t,x) = \chi \bar{\rho}(\tau t, \lambda x)$. Then

$$\partial_t \tilde{\rho} = \tau \chi^{1-\alpha} \lambda^{-2} \Delta \tilde{\rho}^\alpha.$$

Get a one parameter family of scaling invariances

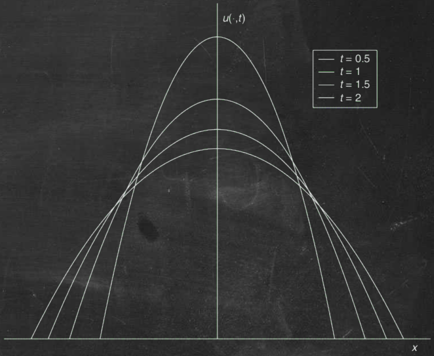
$$\tau \chi^{1-\alpha} \lambda^{-2} = 1.$$

Consider the ZRP with local jump rate function $g(\eta) = \eta^\alpha$, $\alpha \geq 1$.

Rescaling particle sizes by χ_N

$$\mu^N(x,t) := \chi_N \Big(\frac{1}{N} \sum_k \delta_k(x) \eta(k,t) \Big) (xN, t \frac{N^2}{\chi_N^{1-\alpha}}).$$

Macroscopic: Barenblatt solution



Two difficulties:

- Superlinear growth of $g(\eta) = \eta^\alpha$, $\alpha > 1$. Possible concentration of mobility $(\eta^N(x))^\alpha$.
- Degeneracy of $g(\eta) = \eta^\alpha$ at $\eta = 0$. Now becomes visible with χ small. Dirichlet form degenerates.

As a result, the classical one-block, two-block approach to the superexponential replacement lemma is not applicable.

Solution: New microscopic, “pathwise” entropy-dissipation inequality

Theorem (Hydrodynamic limit, G., Heydecker, 2023)

Let $\rho_0 \in L^1_{\geq 0}(\mathbb{T}^d)$ *with finite entropy* $\mathcal{H}(\rho_0) = \int \rho_0 \log \rho_0 < \infty$, $\limsup_{M\to\infty} \limsup_{N\to\infty} \mathbb{P}(\mathcal{H}(\eta_0^N) > M) = 0$ *and*

$$\mathbb{P}(d(\eta_0^N, \rho_0) > \varepsilon) \to 0.$$

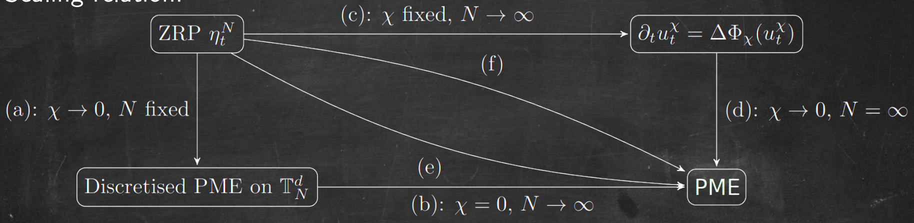
Assume the scaling relation $\chi_N^{1\wedge\alpha/2} \leq CN^{-2}$. *Then*

$$\mu^N(t) \rightharpoonup^* \bar{\rho}(t)\, dx$$

in probability, where $\bar{\rho}$ *is the solution to*

$$\partial_t \bar{\rho} = \Delta \bar{\rho}^\alpha.$$

Scaling relation:



Here: Assume the “scaling relation”

$$\chi_N^{1 \wedge \alpha/2} \leq C N^{-2}.$$

Large deviations around the porous medium equation?

Rate function

$$\mathcal{I}(\rho) = \inf \left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dxds : \ g \in L^2_{t,x}, \ \underbrace{\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g)}_{\text{"skeleton equation"}} \right\}$$

Theorem ([Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995])

For every $, \mathcal{O} \subseteq D([0,T], \mathcal{M}_+)$ *open we have*

$$e^{-N \inf_{\rho \in \mathcal{O}} \overline{\mathcal{I}_{|A}}(\rho)} \lesssim \mathbb{P}[\mu^N \in \mathcal{O}] \lesssim e^{-N \inf_{\rho \in \bar{\mathcal{O}}} \mathcal{I}(\rho)}$$

where A *is the set of nice fluctuations* $\mu = \rho \, dx$ *with* ρ *a solution to*

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g)$$

for some $\underline{g \in C^{1,3}_{t,x}}$. *Problem:* $\mathcal{I} = \overline{\mathcal{I}_{|A}}$?

One approach: Show well-posedness of

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g), \quad \text{with } g \in L^2_{t,x}.$$

Theorem (The skeleton equation, Fehrman, G. 2023)

Let $g \in L^2_{t,x}$, ρ_0 non-negative and $\int \rho_0 \log(\rho_0) dx < \infty$. There is a unique weak solution to

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

The map $g \mapsto \rho$, $L^2_{t,x} \to L^1_{t,x}$, is weak-strong continuous.

Theorem (LDP for zero range process, G., Heydecker, 2023)

The rescaled zero range process satisfies the full large deviations principle with speed $\frac{N^d}{\chi_N}$ and rate function

$$\mathcal{I}(\rho) = \inf \left\{ \|g\|^2_{L^2_{t,x}} : \partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g) \right\}.$$

Gradient flow structures for the porous medium equation

The PME as a gradient flow: What is $\mathcal{H}$ in

$$\rho = \mathrm{argmin}_\rho \mathcal{I}(\rho) = \mathrm{argmin}_\rho \Big(\mathcal{H}(\rho_T) - \mathcal{H}(\rho_0) + \frac{1}{2} \int_0^T \int \frac{|\nabla \rho^{\frac{\alpha+1}{2}}|^2}{\rho} dxdt + \frac{1}{2} \mathcal{A}(\rho) \Big)$$

From LDP we know that

$$\rho = \mathrm{argmin}_\rho \mathcal{I}(\rho)$$

with

$$\mathcal{I}(\rho) = \inf \left\{ \int_0^T \int_{\mathbb{T}} |g|^2 dxds : g \in L^2_{t,x}, \underbrace{\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^{\frac{\alpha}{2}} g)}_{\text{"skeleton equation"}} \right\}.$$

How does this relate? Have to rewrite rate function in terms of entropy.

Informally: If we are able to write $\Delta\rho^\alpha = -\nabla_{\mathcal{M}}\mathcal{H}(\rho^\alpha)$ then we have the following identity

$$
\begin{aligned}
\mathcal{I}(\rho) =& \frac{1}{2}\inf\left\{\int_0^T\int_{\mathbb{T}}|g|^2 dxds : g \in L^2_{t,x},\ \partial_t\rho = \Delta\rho^\alpha + \nabla\cdot(\rho^{\alpha/2}g)\right\} \\
&= \frac{1}{2}\int_0^T \|\partial_t\rho - \Delta\rho^\alpha\|^2_{H^{-1}_{\rho^\alpha}} \\
&= \frac{1}{2}\int_0^T \|\partial_t\rho\|^2 - \int_0^T (\partial_t\rho, -\nabla_{\mathcal{M}}\mathcal{H}(\rho^\alpha))_{H^{-1}_{\rho^\alpha}} + \frac{1}{2}\int_0^T \|\Delta\rho^\alpha\|^2_{H^{-1}_{\rho^\alpha}} \\
&=_{\text{difficult!}} \mathcal{H}(\rho_T) - \mathcal{H}(\rho_0) + \frac{1}{2}\mathcal{A}(\rho) + \frac{1}{2}\|\Delta\rho^\alpha\|^2_{\dot{H}^{-1}_{\rho^\alpha}}.
\end{aligned}
$$

Define the action

$$
\mathcal{A}(\rho) = \inf\{\|g\|^2_{L^2_{t,x}} : \partial_t\rho + \underbrace{\nabla\cdot(\rho^{\alpha/2}g)}_{=M^{\frac{1}{2}}(\rho)} = 0\}.
$$

In conclusion, the gradient flow picture suggests the energy identity

$$
\mathcal{I}(\rho) = \mathcal{H}(\rho_T) - \mathcal{H}(\rho_0) + \frac{1}{2}\mathcal{A}(\rho) + \frac{1}{2}\int_0^T \|\rho^{\alpha/2}\|^2_{\dot{H}^1}.
$$

Theorem (Entropy dissipation equality, G., Heydecker, 2023)

Let $D_\alpha(\rho) < \infty$, $\mathcal{H}(\rho_0) < \infty$, $u_0 > 0$. *Then*

$$\mathcal{I}(\rho) = \mathcal{H}_{u_0}(\rho_T) - \mathcal{H}_{u_0}(\rho_0) + \frac{1}{2}\mathcal{A}(\rho) + \frac{1}{2}\int_0^T \|\rho^{\alpha/2}(s)\|_{\dot{H}^1}^2 ds.$$

If ρ *is a solution to the PME, we have the energy equality*

$$0 = \mathcal{H}_{u_0}(\rho_T) - \mathcal{H}_{u_0}(\rho_0) + \int_0^T \|\rho^{\alpha/2}(s)\|_{\dot{H}^1}^2 ds.$$

Sketch of the proof

In equilibrium, detailed balance $\implies$ $(\mathcal{T}\eta^N_\bullet)_t := \eta^N_{T-t-}$ has the same law as the original process.

Contraction principle and uniqueness of rate functions:

$$\mathcal{I}(\mathcal{T}\rho) = \mathcal{I}(\rho)$$

for all ρ. Analyse identity without assuming any more regularity on ρ than necessary.

Remark

- The same identity as informally suggested in Dirr-Stamatakis-Peletier.
- Sandier-Serfaty (in)equality for the formal Riemannian structure.
- LDP allows us to avoid proving a 'chain rule for entropy' (Erbar, '16).
- Same argument: equality in the *H*-Theorem for (PME):

$$\mathcal{H}(u_t) + \int_0^t \alpha \mathcal{D}_\alpha(u_s) ds = \mathcal{H}(u_0).$$

A new look at properties of the skeleton equation

$$\partial_t \rho = \Delta \rho^\alpha + \nabla \cdot (\rho^\alpha \nabla H) = \Delta \rho^\alpha + \nabla \cdot (\rho^{\alpha/2} g)$$

- Construction of g_{r} shows how *anti*dissipative effects can arise, since

$$\begin{aligned} \partial_t \rho_r &= -\Delta \rho_r^\alpha + \nabla \cdot (\rho_r^{\alpha/2} \mathcal{T} g) \\ &= \Delta \rho_r^\alpha - \nabla \cdot \rho_r^{\frac{\alpha}{2}} (g_r). \end{aligned}$$

- Hence why L^p_x estimates had to be false: trajectories with $\rho_0 \notin L^p_x$, $\rho_T \in C^\infty_x$ give reversal $\rho_0 \in C^\infty_x$ but $\rho_T \notin L^p_x$.

References:

B. Fehrman and B. Gess.
Non-equilibrium large deviations and parabolic-hyperbolic PDE with irregular drift.

B. Gess and D. Heydecker.
The Porous Medium Equation: Large Deviations and Gradient Flow with Degenerate and Unbounded Diffusion.