Regularization and well-posedness by noise

Benjamin Gess
Max Planck Institute for Mathematics in the Sciences, Leipzig
& Universität Bielefeld

Nonlinear Stochastic Evolution Equations: Analysis and Numerics
TU Berlin, November 2016

joint work with: Paul Gassiat, Mario Maurelli
[Gassiat, G.; arxiv],[G., Maurelli.; ongoing].

Outline

1. Introduction
2. Well-posedness by noise for stochastic scalar conservation laws
3. Regularization by noise for stochastic Hamilton-Jacobi equations

Introduction

- General aim: Regularization or well-posedness by inclusion of stochastic perturbations
- Problem: A-priori unclear which form of noise to consider
- Additive noise, e.g.

$$du = \Delta u\, dt + f(u)\, dt + dW_t \quad \text{[Gyöngy, Pardoux; 1993]}$$

$$du + (u \cdot \nabla) u\, dt + \nabla p\, dt = \Delta u\, dt + dW_t \quad \text{[Flandoli, Romito; 2007].}$$

- More recent: Linear multiplicative noise

$$du + b(x) \cdot \nabla u\, dt = \nabla u \circ d\beta_t. \quad \text{[Flandoli, Gubinelli, Priola; 2010].}$$

Introduction

- We recall: Consider
$$\partial_t u + b(x) \cdot \nabla u = 0, \tag{TE}$$
for non-Lipschitz b (but, say, Hölder continuous). E.g. $b(x) = \mathrm{sgn}(x)\sqrt{|x|}$.
- In general, characteristics collide causing shocks (i.e. discontinuities). Solution is not better than $u(t) \in BV$ even if u_0 is smooth.
- Characteristics branch causing non-uniqueness of weak solutions.
- Question: Can noise restore uniqueness or increase regularity?

Introduction

- Consider
$$du + b(x)\cdot\nabla u = \nabla u \circ d\beta_t. \tag{STE}$$
- Two (related) effects: regularization by noise, well-posedness by noise.
- Well-posedness by noise [Flandoli, Gubinelli, Priola; 2010]: Weak solutions to (STE) are unique.
- Regularization by noise [Flandoli, Fedrizzi; 2013]: If u_0 is smooth then $u(t)$ is smooth.

Introduction

- Left open: What about the nonlinear case, e.g. Burgers?
- Linear multiplicative noise does not help anymore:

$$du + \partial_x u^2 \, dt = \partial_x u \circ d\beta_t.$$

- Then: $v(t,x) := u(t, x - \beta_t)$ is the unique solution to

$$\partial_t v + \partial_x v^2 = 0.$$

- In particular, weak solutions are non-unique.

Well-posedness by noise for stochastic scalar conservation laws

Well-posedness by noise for stochastic scalar conservation laws

Introduction

Consider

$$\partial_t u + b(x) \cdot \nabla(u^2) = 0,$$

for irregular b (in particular $\mathrm{div} b \notin L^\infty$):

- The deterministic problem is ill-posed in general (entropy solutions are non-unique)
- Can we restore well-posedness by adding a linear multiplicative noise term?
- Non-trivial: shocks due to the nonlinearity and shocks due to the irregularity of b may combine in such a way that this noise may be insufficient.

Stochastic Burgers' equation:

$$du + b(x) \cdot \nabla(u^2)dt = \nabla u \circ d\beta_t.$$

Theorem

Assume $b \in (W^{1,1} \cap L^\infty)(\mathbb{R}^d)$ and $\operatorname{div} b \in L^p(\mathbb{R}^d)$ for some $p > d$. Then the stochastic Burgers' equation admits a unique entropy solution.

Model example: $b(x) = \operatorname{sgn}(x)|x|^{1/2}$.

Given $u = u(t,\omega,x)$, introduce a new (velocity) variable $\xi \in \mathbb{R}$ and define the kinetic function:

$$f = f[u](t,\omega,x,\xi) = 1_{0<u(t,\omega,x)<\xi}.$$

Definition

u is an entropy solution if $f[u]$ is adapted and solves (in the sense of distributions) the kinetic equation

$$\partial_t f + 2b\xi \cdot \nabla_x f + \nabla_x f \circ d\beta_t = \partial_\xi m$$

for some nonnegative random measure m on $[0,T] \times \mathbb{R}^d_x \times \mathbb{R}_\xi$.

Definition

$f = f(t,\omega,x,\xi)$ is a generalized entropy solution if f solves the kinetic equation, $|f| = \mathrm{sgn}(\xi) f \leq 1$ and $\partial_\xi f = \delta_0 - \nu$, ν positive random measure.

- Method to solve the equation:
  - Existence of a generalized entropy solution (standard, valid for general b)
  - Every generalized solution is an entropy solution, equivalently $|f| - f^2 = 0$.
- Generalized kinetic solutions are entropy solutions:
  - First step (deterministic): Via renormalization arguments derive an inequality for $|f| - f^2$ (similar to the kinetic equation).
  - Second step (stochastic): Take the expectation and use parabolic theory.
- Kinetic equation in Itô form:

$$\partial_t f + 2b\xi \cdot \nabla_x f + \nabla_x f \, d\beta_t - \frac{1}{2}\Delta_x f = \partial_\xi m$$

A Laplacian appears, which suggests regularization. Note: the equation is hyperbolic (not parabolic: no regularization of initial datum).

Renormalization step:

- By informal computations, $|f| - f^2$ satisfies

$$\partial_t(|f| - f^2) + 2b\xi \cdot \nabla_x(|f| - f^2) + \nabla_x(|f| - f^2) \circ d\beta_t = (\mathrm{sgn}(\xi) - 2f)\partial_\xi m$$

- Rigorous: Need $b \in W^{1,1}$ for commutator estimates [DiPerna-Lions 89, Ambrosio 04].
- Using $\partial_\xi f = \delta_0 - \nu$, $\partial_\xi(\mathrm{sgn}(\xi)) = 2\delta_0$ and integration by parts for φ independent of ξ, we get

$$\int_{x,\xi} (\mathrm{sgn}(\xi) - 2f)\partial_\xi m \varphi dx d\xi = -2\int_{x,\xi} \nu m \varphi dx d\xi \le 0$$

  Thus,

$$\partial_t(|f| - f^2) + 2b\xi \cdot \nabla_x(|f| - f^2) + \nabla_x(|f| - f^2) \circ d\beta_t \le 0.$$

Corollary

With the previous assumptions,

$$\partial_t(E[|f|-f^2],\varphi) \leq (E[|f|-f^2],\partial_t\varphi - 2div_x(b\xi\varphi)+\frac{1}{2}\Delta_x\varphi).$$

Proposition

Fix $T>0$. There exists $\varphi \geq 0$, independent of ξ, with $\varphi_T \sim 1$, such that

$$\partial_t\varphi - 2div_x(b\xi\varphi)+\frac{1}{2}\Delta_x\varphi \leq C$$

for some $C>0$ (independent of T).

Corollary

$|f|-f^2=0$ *and so the main result follows.*

Proof of the proposition in two steps. Recall

$$\partial_t \varphi - 2\mathrm{div}_x(b\xi\varphi) + \frac{1}{2}\Delta_x \varphi \leq C$$

Step 1: Take $\varphi \geq 0$ solution to

$$\partial_t \varphi - 2R|\mathrm{div}_x b|\varphi + \frac{1}{2}\Delta_x \varphi = 0$$

Lemma

Assume divb in L^p for some $p > d$. Then φ is in $W^{1,\infty}(\mathbb{R}^d)$ (uniformly in time).

Proof of the lemma based on heat kernel estimates (here forward equation):

$$\varphi_t = P_t \varphi_0 + \int_0^t P_{t-s}(2R|\mathrm{div} b|\varphi_s)ds$$

Control of $\|\nabla \varphi\|_{L_x^\infty}$ needed for step 2.

Step 2: Conclusion:

$$
\begin{aligned}
&\partial_t \varphi - 2\mathrm{div}_x(b\xi\varphi) + \frac{1}{2}\Delta_x \varphi \\
&\le (\partial_t \varphi - 2R|\mathrm{div}_x b|\varphi + \frac{1}{2}\Delta_x \varphi) \\
&\quad + 2(\xi \mathrm{div}_x b - R|\mathrm{div}_x b|)\varphi \\
&\quad + b \cdot \nabla_x \varphi \\
&\le \|b\|_{L^\infty} \|\varphi\|_{L^\infty} =: C
\end{aligned}
$$

Regularization by noise for stochastic Hamilton-Jacobi equations

Regularization by noise for stochastic Hamilton-Jacobi equations

Introduction

- Can we use nonlinear noise to regularize nonlinear PDE?
- Model example: Porous medium equation

$$\partial_t w = \frac{1}{6}\partial_{xx} w^3, \quad \text{on } \mathbb{R},$$

  with smooth, compactly supported initial condition.
- Solutions are known to develop singularities:

$$\|\partial_x w(t)\|_{L^\infty} = \infty,$$

  for all $t > 0$ large enough.
- Linear multiplicative noise does not help:

$$dv = \frac{1}{6}\partial_{xx} v^3 \, dt + \sigma \partial_x v \circ d\beta_t, \quad \text{on } \mathbb{R}.$$

  Then $w(t,x) = v(t, x - \sigma\beta_t)$.

Introduction

- Instead, consider, for $\sigma > 0$,

$$dv = \frac{1}{6}\partial_{xx} v^3 \, dt + \sigma \frac{1}{2}\partial_x v^2 \circ d\beta_t, \quad \text{on } \mathbb{R}. \tag{SPME}$$

- Note: If u is the viscosity solution to

$$du = \frac{1}{6}\partial_x(\partial_x u)^3 \, dt + \sigma \frac{1}{2}(\partial_x u)^2 \circ d\beta_t, \quad \text{on } \mathbb{R},$$

then, $v = \partial_x u$ solves (SPME).

Setup

- General framework: Consider

$$du = F(t,x,u,Du,D^2u)\,dt + \frac{1}{2}|Du|^2 \circ d\beta_t, \quad \text{on } \mathbb{R}^N,$$

  and compare to the regularity of the non-perturbed problem

$$dw = F(t,x,w,Dw,D^2w), \quad \text{on } \mathbb{R}^N.$$

- F satisfies the usual assumptions from the theory of stochastic viscosity solutions

Key result

- Control on the rate of loss of semiconcavity: There is a $V_F \in Lip_{loc}(\mathbb{R}_+ \setminus \{0\})$ such that, for $\ell_0 > 0$,

$$D^2 g \leq \frac{Id}{\ell_0} \;\Rightarrow\; D^2(S_F(t,g)) \leq \frac{Id}{\ell(t)},$$

where $t \mapsto S_F(t,g)$ denotes the solution to

$$\begin{aligned} \partial_t w + F(t,x,w,Dw,D^2 w) &= 0 \\ w(0) &= g \end{aligned}$$

and ℓ the solution to

$$\begin{aligned} \dot{\ell}(t) &= V_F(\ell(t)) \\ \ell(0) &= \ell_0. \end{aligned}$$

Key result

Theorem

Let u be the solution to

$$\partial_t u = F(t,x,u,Du,D^2u) + \frac{1}{2}|Du|^2 \circ d\beta_t,$$

with

$$D^2 u_0 \le \frac{Id}{\ell_0} \quad \text{some } \ell_0 \in [0,\infty).$$

Then for each $t \ge 0$, one has

$$D^2 u(t,\cdot) \le \frac{Id}{L_t}, \tag{1}$$

where L is the maximal solution to

$$\begin{aligned} dL_t &= V_F(L_t)dt + d\beta_t, \ \text{ with } L_t \ge 0, \\ L_0 &= \ell_0. \end{aligned} \tag{2}$$

Model example

- Return to the model example

$$dv = \frac{1}{6}\partial_{xx} v^3\, dt + \sigma \frac{1}{2}\partial_x v^2 \circ d\beta_t, \quad \text{on } \mathbb{R}.$$

- Deterministic case: $\|\partial_x w(t)\|_\infty = \infty$ for all $t > 0$ large enough.
- We have the *sharp* bound

$$\|\partial_x v(t)\|_\infty \leq \frac{1}{L^+(t) \wedge L^-(t)},$$

where $L^\pm$ solve

$$dL^\pm = -\frac{1}{L^\pm(t)}\, dt \pm \sigma d\beta_t, \text{ with } L_t^\pm \geq 0,$$
$$L^\pm(0) = \frac{1}{\|(\partial_x v_0)_\pm\|_\infty}.$$

Model example

- In conclusion,
  - If $\sigma^2 > 2$: For all $t \geq 0$, $\mathbb{P}$-a.s.

$$v(t) \in W^{1,\infty}$$

  - If $\sigma^2 \leq 2$: $\mathbb{P}$-a.s. for all $t > 0$ large enough

$$v(t) \notin W^{1,\infty}$$

Thanks

Thanks!