Optimal regularity for the porous medium equation

Benjamin Gess
Max Planck Institute for Mathematics in the Sciences, Leipzig
& Universität Bielefeld

Universität Bielefeld
Bielefeld - Edinburgh - Swansea Stochastic Spring
March 2018

[G.; arxiv, 2017].

Outline

- We consider the porous medium equation

$$\partial_t u = \Delta u^{[m]} \text{ on } (0,T)\times\mathbb{R}^d$$
$$u(0) = u_0 \text{ on } \mathbb{R}^d,$$

  with $u_0 \in L^1(\mathbb{R}^d)$, $m > 1$.
- Aim: Optimal regularity of solutions in (fractional) Sobolev spaces.

Introduction - Applications

Introduction - Applications

Application 1: Flow of ideal gas in a homogeneous porous medium.

- Let ρ be the density, p be the pressure and V be the velocity of the (macroscopic) gas.
- Mass balance:

$$\varepsilon \partial_t \rho + \mathrm{div}(\rho V) = 0.$$

- Darcy's law:

$$\mu V = -k \nabla p.$$

- State equation, for perfect gases:

$$p = p_0 \rho^{\gamma},$$

e.g. air at normal temperature $\gamma = 1.405$.

- Combining the constitutive equations yields

$$\partial_t \rho = c \Delta(\rho^m)$$

with $m = 1 + \gamma$ and $c = \frac{\gamma k p_0}{(\gamma+1)\varepsilon\mu}$.

Application 2: Population dynamics

- Spreading of biological populations

$$\partial_t u = \operatorname{div}(\kappa \nabla u) + f(u),$$

  where u is the density of the species, $f(u)$ is the reproduction/death rate.
- If populations avoid crowding κ is an increasing function of the population density, $\kappa = \varphi(u)$ with φ increasing.
- In particular cases we have $\varphi(u) = au$. Hence,

$$\partial_t u = \frac{a}{2}\Delta u^2 + f(u),$$

- Random environment leads to the degenerate parabolic Anderson model

$$\partial_t u = \frac{a}{2}\Delta u^2 + u\xi,$$

  where ξ is spatial white noise.

Application 3: Interacting particles

- Interacting particle system

$$\frac{d}{dt}X_t^i = -\frac{1}{L}\sum_{j=1,j\neq i}^{L} \nabla V_L(X_t^i - X_t^j) \quad i = 1\dots L, \tag{1}$$

  where V_L is a rescaled interaction potential

$$V_L(x) = \lambda^d V_1(\lambda x), \lambda = L^{\frac{\beta}{d}}$$

  and $\beta \in (0,1)$.

- Consider the empirical process

$$t \mapsto \mu_t^L = \frac{1}{L}\sum_{i=1}^{L} \delta_{X_t^i}.$$

Under regularity, decay and symmetry assumptions on V_1 obtain

Theorem (Oelschläger)

Suppose that μ_0^L converges for $L \to \infty$ to a function $m_0(x)$. Then μ_t^L converges to the Boussinesq equation

$$\partial_t m = c\Delta m^2$$

with initial data m_0, $c = \frac{1}{2}\int V_1(x)\,dx$.

Application 4: Mean field interacting particles

- Mean-field interacting particles: L randomly moving particles with diffusivity depending on the (local) concentration

$$dX_t^i = \sigma^L(X_t^i, \underbrace{\frac{1}{L}\sum_{j\neq i}\delta_{X_t^j}}_{=:\mu_L})dW_t^i \text{ for } t \geq 0 \text{ and } i \in \{0,\dots,L\}.$$

- Informally: Let number of particles $L \to \infty$ and rescale space $\sigma^L(x, m(x)dx) \to \sigma(m(x))$. Then, informally, $\mu_L \rightharpoonup^* m\,dx$ with m solving

$$dm = \frac{1}{2}\Delta(\sigma^2(m)m).$$

- Further applications
  - Nonlinear heat transfer
  - Continuum limit of zero range processes
  - Groundwater flow. Boussinesq's equation
  - Thin liquid film spreading under gravity (neglecting surface tension)

Scaling arguments and special solutions

Scaling arguments and special solutions

- Note

$$\partial_t u = \Delta u^{[m]} = m\mathrm{div}(|u|^{m-1}\nabla u)$$
$$= m|u|^{m-1}\Delta u + m(m-1)u^{[m-2]}|\nabla u|^2.$$

- Barenblatt solution:

$$U(x,t) = t^{-\alpha}F(xt^{-\beta}) = t^{-\alpha}(C - k|xt^{-\alpha/d}|^2)_+^{\frac{1}{m-1}},$$

where $\alpha = \frac{d}{d(m-1)+2}$, $k = \frac{(m-1)\alpha}{2md}$. We observe that

$$lim_{t\downarrow 0} U(x,t) = M\delta_0(x)$$

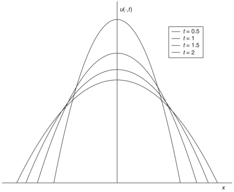

Figure: Fundamental solution of the porous medium equation

Lemma

Assume that for some $s \geq 0$, $p \geq 1$, $C \geq 0$ we have

$$\|u\|^p_{L^p([0,T];\dot{W}^{s,p}(\mathbb{R}^d_x))} \leq C\|u_0\|_{L^1(\mathbb{R}^d_x)},$$

for all solutions u to PME. Then, necessarily $p \leq m$ and $s \leq \frac{2}{m}$.

Example

Consider the Barenblatt solution

$$u(t,x) = t^{-\alpha}(C - k|xt^{-\beta}|^2)_+^{\frac{1}{m-1}}.$$

Then

$$u \in L^m([0,T]; \dot{W}^{s,m}(\mathbb{R}^d_x))$$

implies $s < \frac{2}{m}$.

Proof.

With $F(x) = (C - k|x|^2)_+^{\frac{1}{m-1}}$ we have $u(t,x) = t^{-\alpha} F(xt^{-\beta})$. We next observe that, for $s \in (0,1)$,

$$\begin{aligned}\|u(t,\cdot)\|^m_{\dot{W}^{s,m}(\mathbb{R}^d_x)} &= \int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_y} \frac{|u(t,x)-u(t,y)|^m}{|x-y|^{sm+d}} dxdy \\ &= t^{-\alpha m - \beta(sm+d)+2d\beta} \|F\|^m_{\dot{W}^{s,m}(\mathbb{R}^d_x)}.\end{aligned}$$

Hence,

$$\|u\|^m_{L^m([0,T];\dot{W}^{s,m}(\mathbb{R}^d_x))} = \|t^{-\alpha m - \beta(sm+d)+2d\beta}\|_{L^1([0,T])} \|F\|^m_{\dot{W}^{s,m}(\mathbb{R}^d_x)}.$$

which is finite if and only if

$$-\alpha m - \beta(sm+d) + 2d\beta > -1 \quad \text{and} \quad F \in \dot{W}^{s,m}(\mathbb{R}^d_x).$$

This is equivalent to $2 > ms$. □

Existing regularity results

Existing regularity results

Let $\dot{\mathcal{N}}^{s,p}$ be the homogeneous Nikolskii space ($\dot{\mathcal{N}}^{s,p} = \dot{B}^s_{p,\infty}$).

Theorem (Ebmeyer, Tadmor, Tao)

Let $u_0 \in L^2(\mathbb{R}^d_x)$. Then

$$\|u\|^{m+1}_{L^{m+1}([0,T];\dot{\mathcal{N}}^{\frac{2}{m+1},m+1}(\mathbb{R}^d_x))} \leq C_m \|u_0\|^2_{L^2_x}.$$

- Note: $\frac{2}{m+1} \leq 1$, which is inconsistent with the linear case $(m=1)$ and with the optimal regularity of the Barenblatt solution.

Consider

$$\partial_t u = \Delta u^{[m]} + S.$$

By (soft) energy methods may be improved to:

Theorem (G. 2017)

Let $\varepsilon > 0$, $m \geq 2$ and $u_0 \in L^{1+\varepsilon}(\mathbb{R}^d_x)$, $S \in L^{1+\varepsilon}([0,T] \times \mathbb{R}^d_x)$. Then

$$\|u\|^{m+\varepsilon}_{L^{m+\varepsilon}([0,T];\dot{\mathcal{N}}^{\frac{2}{m+\varepsilon},m+\varepsilon}(\mathbb{R}^d_x))} \leq C_{\varepsilon,m}\|u_0\|^{1+\varepsilon}_{L^{1+\varepsilon}_x}.$$

- Note: optimal regularity for the Barenblatt solution, but $m \geq 2$ implies $\frac{2}{m+\varepsilon} < 1$.
- Problem: How to get to more than one derivative?

Optimal regularity for the porous medium equation

Optimal regularity for the porous medium equation

Consider

$$\partial_t u = \Delta u^{[m]} + S(t,x) \quad \text{on } (0,T)\times\mathbb{R}^d_x \tag{PME}$$

with $u_0 \in L^1(\mathbb{R}^d_x)$, $S \in L^1([0,T]\times\mathbb{R}^d_x)$.

Theorem (G., 2017)

Let $u_0 \in L^1(\mathbb{R}^d_x)$, $S \in L^1([0,T]\times\mathbb{R}^d_x)$. Let u be the unique entropy solution to the PME. Then, for all

$$s \in [0,\frac{2}{m}), \quad p \in [1,m)$$

we have

$$u \in L^p([0,T];\dot{W}^{s,p}_{loc}(\mathbb{R}^d_x)).$$

In addition, for all $\mathcal{O} \subset\subset \mathbb{R}^d$ there is a constant $C = C(m,p,s,\varepsilon,T,\mathcal{O})$ such that

$$\|u\|_{L^p([0,T];\dot{W}^{s,p}(\mathcal{O}))} \le C\left(\|u_0\|^2_{L^1_x}+1\right).$$

Elements of the proof (for simplicity $S = 0$)

- Kinetic form: Introduce

$$f = \chi(u(t,x), v) = 1_{v<u(t,x)} - 1_{v<0}.$$

  Then,

$$\partial_t f = m|v|^{m-1}\Delta f + \partial_v q \text{ on } (0,T)\times\mathbb{R}^d_x\times\mathbb{R}_v,$$

  for some $q \in \mathscr{M}^+$.
- Fourier transformation in time and space (modulo cut-off in time)

$$\underbrace{i\tau\hat{f} - m|v|^{m-1}|\xi|^2\hat{f}}_{=:\mathscr{L}(i\tau,\xi,v)\hat{f}} = \partial_v\hat{q}.$$

- Hence, informally,

$$\hat{f} = \frac{1}{i\tau - m|v|^{m-1}|\xi|^2}\partial_v\hat{q} = \frac{1}{\mathscr{L}(i\tau,\xi,v)}\partial_v\hat{q}.$$

- Gain regularity, depending on the degeneracy of the operator $\mathscr{L}(i\tau,\xi,v)$ in $v = 0$.
- Idea: Micro-local decomposition of the Fourier-space depending on the degeneracy in $|v|^{m-1}$.

Obstacles:

1. Bootstrapping: Established methods rely on bootstrapping, i.e. assuming that $u \in W_x^{\alpha,1}$ for some α use that $f = \chi(u) \in W_{x,v}^{\alpha,1}$. But: This is true for $\alpha \leq 1$ only!
2. Established methods can only make use of the fact that q has finite mass. This necessarily leads to sub-optimal estimates.
   –> Solution: Use that q allows singular moments $\int |v|^{-1+} dq < \infty$.
3. Integrability: Established methods yield good estimates only in an L^2-framework. This prevents from obtaining optimal integrability exponents
   –> Introduce a new notion of isentropic truncation properties for Fourier multipliers.

Optimal regularity for the degenerate parabolic Anderson model

Optimal regularity for the degenerate parabolic Anderson model

- The degenerate parabolic Anderson model:

$$\partial_t u = \Delta u^{[m]} + u\xi \quad \text{on } (0,T)\times I$$

with $u_0 \in L^1(I)$, ξ spatial white noise, $I \subseteq \mathbb{R}$ bounded interval and zero Dirichlet boundary conditions.

- Note: $\xi \in C^{-1/2-} = B^{-1/2-}_{\infty,\infty}$.

Corollary

Let $u_0 \in L^{m+1}(I)$. Then there exists a weak solution u satisfying, for all $p \in [1,m)$, $s \in [0, \frac{3}{2}\frac{1}{m})$,

$$u \in L^p([0,T]; W^{s,p}_{loc}(I)),$$

with, for all $T \geq 0$, $\mathscr{O} \subset\subset I$,

$$\|u\|_{L^p([0,T];W^{s,p}(\mathscr{O}))} \lesssim \|u_0\|^{m+1}_{L^{m+1}(I)} + \|S\|^{\tau}_{B^{-\eta}_{\infty,\infty}} + 1,$$

for some $\tau \geq 2$ and $\eta \in (\frac{1}{2}, 1]$ small enough.

Thanks

Thanks!