Stochastic scalar conservation laws and Hamilton-Jacobi equations

Benjamin Gess
Max Planck Institute for Mathematics in the Sciences, Leipzig
& Universität Bielefeld

C.I.R.M., 23-26 April, 2019
Random Partial Differential Equations

[G., Souganidis.; CPAM 2018], [Gassiat, G., Lions, Souganidis.; arxiv 2018], [Gassiat, G.; PTRF 2018].

Outline

- Stochastic Hamilton-Jacobi equations

$$du = F(t,x,u,Du,D^2u)\,dt + \sum_i H_i(x,Du)\circ d\beta_t^i.$$

E.g. stochastic mean curvature flow [Kawasaki, Ohta (1982)].
Model case

$$du = \partial_x(\partial_x u)^3\,dt + \sigma(\partial_x u)^2 \circ d\beta_t.$$

- Stochastic scalar conservation laws

$$du + \underbrace{\mathrm{div}(H(x,u)\circ d\beta)}_{\sum_{i,j}\partial_{x_i}H^{i,j}(x,u)\circ d\beta_t^j} = 0.$$

E.g. particle systems with common noise, mean-field games.
Model case

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ d\beta_t = 0.$$

Regularization by noise for stochastic conservation laws

Consider

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ d\beta_t = 0 \quad \text{on } (0,T)\times\mathbb{T}$$
$$u_0 \in L^\infty(\mathbb{T})$$

What is the regularity of solutions (β_t Brownian motion) vs. the deterministic case ($\beta_t = t$)?

Consider

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0, \quad \text{on } (0,T)\times\mathbb{R}$$
$$u(0) = u_0 \in L^\infty(\mathbb{R}^d).$$

For

$$\chi(t,x,v) = \chi(u(t,x),v) = 1_{v<u(t,x)} - 1_{v<0}$$

we get the kinetic form

$$\partial_t\chi + v\partial_x\chi = \partial_v m \quad \text{on } (0,T)\times\mathbb{R}\times\mathbb{R}.$$

Dissipation-dispersion approximations lead to

Definition (De Lellis, Otto, Westdickenberg, 2003)

A function $u \in L^\infty([0,T]\times\mathbb{R})$ is said to be a quasi-solution if $\chi(t,x,v) = \chi(u(t,x),v)$ satisfies

$$\partial_t\chi + v\partial_x\chi = \partial_v m \quad \text{on } (0,T)\times\mathbb{R}\times\mathbb{R}$$

for some finite (signed) measure m.

Theorem (De Lellis, Westdickenberg, 2003; Jabin, Perthame 2002)

Consider

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0, \quad \text{on } (0,T)\times\mathbb{R}.$$

Then

1. *Each quasi-solution satisfies, for all* $\lambda \in (0, \frac{1}{3})$,

$$u \in L^1([0,T]; W^{\lambda,1}(\mathbb{R})).$$

2. *For each* $\lambda > \frac{1}{3}$ *there exists a quasi-solution* u, *such that* u *is a weak solution and*

$$u \notin L^1([0,T]; W^{\lambda,1}(\mathbb{R})).$$

Theorem

Let β^H be a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2}]$ and u be a bounded quasi-solution to

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ d\beta_t^H = 0 \quad \text{on } \mathbb{T}. \tag{1}$$

Then, for all $\lambda < \frac{1}{1+2H}$, $\mathbb{P}$-a.s.,

$$u \in L^1([0,T]; W^{\lambda,1}(\mathbb{T})).$$

- Consider
$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0 \quad \text{on } (0,T)\times\mathbb{T}.$$
- Kinetic formulation:
$$d\chi + v\partial_x\chi \circ dw_t = \partial_v m,$$
for some finite Radon measure m.
- Change of variables gives
$$\chi(t,x,v) = \chi_0(x+vw_t,v) + \int_0^t \partial_v m(s,x+v(w_t-w_s),v)ds.$$
- Averaging over velocity
$$u(t,x) = \int_v \chi = \int_v \chi_0(x+vw_t,v)dv + \int_0^t\int_v \partial_v m(s,x+v(w_t-w_s),v)dvds.$$
- The averaging effect appears since the velocity average in v contains averaging of the x-variable.

- Rigorously, this can be seen by Fourier transform, that is,

$$\hat{u}(t,n)=\int_v e^{-ivw_t n}\hat{\chi}_0(n,v)dv+\int_0^t\int_v e^{-iv(w_t-w_s)n}\partial_v\hat{m}(s,n,v)dvds.$$

- The oscillatory integrals have a regularizing effect, both in v and in w_t-w_s.
- A path $w\in C(\mathbb{R}_+;\mathbb{R}^d)$ is said to be (ρ,γ)-irregular if

$$|\int_s^t e^{iw_r\cdot n}dr|\lesssim(1+|n|)^{-\rho}|t-s|^{\gamma}\quad\forall n\in\mathbb{R}^d,s<t.$$

  [Catellier, Gubinelli; *SPA*, 2016]
- Note:

$$\int_s^t e^{iw_r\cdot n}dr=\int_{\mathbb{R}}e^{ix\cdot n}dL_w^{s,t}(x)=\hat{L_w^{s,t}}(n)$$

  the Fourier transform of the local time.

Theorem

Let $w \in C^\eta([0,T],\mathbb{R}^d)$ for some $\eta > 0$ be (ρ,γ)-irregular, u a bounded quasi-solution solution to

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ dw_t = 0 \quad \text{on } (0,T)\times\mathbb{T}.$$

Then, for all

$$\lambda < \frac{\rho(\eta+1)-(1-\gamma)}{(\rho\vee 1)(\eta+1)+(1-\gamma)},$$

we have

$$u \in L^1([0,T];W^{\lambda,1}(\mathbb{T})).$$

Application to fractional Brownian motion $\rightarrow$ Theorem above.
Q: What happens when the noise competes with a deregularizing deterministic part?

Regularization by noise for stochastic Hamilton-Jacobi equations

- Consider
$$du = \frac{1}{6}\partial_x(\partial_x u)^3\,dt + \sigma\frac{1}{2}(\partial_x u)^2 \circ d\beta_t.$$
What is the regularity?
- Note: For $v = \partial_x u$ have
$$dv = \frac{1}{6}\partial_{xx} v^3\,dt + \sigma\frac{1}{2}\partial_x v^2 \circ d\beta_t.$$

- Deterministic part

$$\partial_t w = \frac{1}{6}\partial_x(\partial_x w)^3$$

with smooth, compactly supported initial condition.

- Solutions are known to develop singularities:

$$\|\partial_{xx} w(t)\|_{L^\infty} = \infty$$

for all $t > 0$ large enough.

- General framework: Consider

$$du = F(t,x,u,Du,D^2u)\,dt + \frac{1}{2}|Du|^2 \circ d\beta_t, \quad \text{on } \mathbb{R}^N,$$

and compare to the regularity of non-perturbed problem

$$dw = F(t,x,w,Dw,D^2w), \quad \text{on } \mathbb{R}^N.$$

- F satisfies the usual assumptions from the theory of stochastic viscosity solutions

- Control on the rate of loss of semiconcavity: There is a $V_F \in Lip_{loc}(\mathbb{R}_+ \setminus \{0\})$ such that, for $\ell_0 > 0$,

$$D^2 g \leq \frac{Id}{\ell_0} \;\Rightarrow\; D^2(S_F(t,g)) \leq \frac{Id}{\ell(t)},$$

where $t \mapsto S_F(t,g)$ denotes the solution to

$$\begin{aligned} \partial_t w + F(t,x,w,Dw,D^2 w) &= 0 \\ w(0) &= g \end{aligned}$$

and ℓ the solution to

$$\begin{aligned} \dot{\ell}(t) &= V_F(\ell(t)) \\ \ell(0) &= \ell_0. \end{aligned}$$

For $p - Laplace$ $(p = 3)$:

$$V_F(\ell) = -\frac{1}{\ell}.$$

Theorem

Let u be the solution to

$$\partial_t u = F(t,x,u,Du,D^2u) + \frac{1}{2}|Du|^2 \circ d\beta_t,$$

with

$$D^2 u_0 \leq \frac{Id}{\ell_0} \quad \text{some } \ell_0 \in [0,\infty).$$

Then for each $t \geq 0$, one has

$$D^2 u(t,\cdot) \leq \frac{Id}{L_t},$$

where L is the maximal solution to

$$\begin{aligned} dL_t &= V_F(L_t)dt + d\beta_t, \ \text{with } L_t \geq 0, \\ L_0 &= \ell_0. \end{aligned}$$

- Return to the model example

$$du = \frac{1}{6}\partial_x(\partial_x u)^3\, dt + \sigma\frac{1}{2}(\partial_x u)^2 \circ d\beta_t.$$

- Determinstic case: $\|\partial_{xx} w(t)\|_\infty = \infty$ for all $t > 0$ large enough.
- We have the sharp bound

$$\|\partial_{xx} u(t)\|_\infty = \frac{1}{L^+(t) \wedge L^-(t)},$$

where $L^\pm$ solve

$$dL^\pm = -\frac{1}{L^\pm(t)}\, dt \pm \sigma\, d\beta_t, \ \text{ with } L^\pm_t \geq 0,$$
$$L^\pm(0) = \frac{1}{\|(\partial_{xx} u_0)_\pm\|_\infty}.$$

- In conclusion,
  - If $\sigma^2 > 2$: For all $t \geq 0$, $\mathbb{P}$-a.s.

$$u(t) \in W^{2,\infty}$$

  - If $\sigma^2 \leq 2$: $\mathbb{P}$-a.s. for all $t > 0$ large enough

$$u(t) \notin W^{2,\infty}$$

- In conclusion,
  - If $\sigma^2 > 2$: For all $t \geq 0$, $\mathbb{P}$-a.s.
    $$u(t) \in W^{2,\infty}$$
  - If $\sigma^2 \leq 2$: $\mathbb{P}$-a.s. for all $t > 0$ large enough
    $$u(t) \notin W^{2,\infty}$$
- Observation: For
  $$du = H(Du, x) \circ d\beta$$
  dynamics are reversible if $u(t) \in W^{2,\infty}$.

Speed of propagation for stochastic Hamilton-Jacobi equations

- Consider

$$du = H(Du, x) \cdot dw \quad \text{in} \quad \mathbb{R}^d \times (0,T] \qquad u(\cdot,0) = u_0 \quad \text{in} \quad \mathbb{R}^d,$$

with $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ convex and Lipschitz continuous in the first argument, $w \in C_0([0,T])$.

- Given $T > 0$ let

$$\rho_H(w,T) := \sup\Big\{ R \geq 0 : \text{ there exist solutions } u^1, u^2 \text{ and } x \in \mathbb{R}^d, \\ \text{such that } u^1(\cdot,0) = u^2(\cdot,0) \text{ in } B_R(x) \text{ and } u^1(x,T) \neq u^2(x,T) \Big\},$$

where $B_R(x)$ is the ball in $\mathbb{R}^d$ centered at x with radius R.

- Classical (e.g. Crandall, Lions): If w is a BV-path, then

$$\rho_H(w,T) \leq L \|w\|_{TV([0,T])},$$

where L is the Lipschitz constant of H.

- It is easy to see that this is sharp when $\dot{w} \equiv 1$.

What is known:

- General continuous signal w: If $H(p,x) = H_1(p) - H_2(p)$, where H_1, H_2 convex, Lipschitz with Lipschitz constant L and $H_1(0) = H_2(0) = 0$, then, for any constant A, if
$$u(0,\cdot) \equiv A \text{ on } B_R(0),$$
then
$$u(t,\cdot) \equiv A \text{ on } B_{R(t)}(0), \quad \text{for } R(t) := R - L(\max_{s\in[0,t]} w(s) - \min_{s\in[0,t]} w(s)).$$
- This does not imply a finite domain of dependence.
- In fact: For $H(p) = |p_1| - |p_2|$ equality is attained in
$$\rho_H(w,T) \leq L\|w\|_{TV([0,T])},$$
for all continuous w [Gassiat, 2017].
- No finite domain of dependence if $w \notin BV([0,T])$

Conclusion: Consider H convex.

Given $w \in C_0([0,T])$, the sequence $(\tau_i)_{i\in\mathbb{Z}}$ of successive extrema of w:

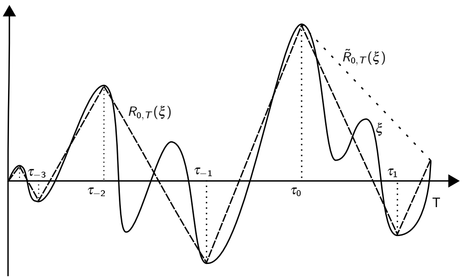


Figure: The (fully) reduced path

Definition

(i) The reduced path $R_{0,T}(w)$ is a piecewise linear function which agrees with w on $(\tau_i)_{i\in\mathbb{Z}}$.
(ii) The fully reduced path $\tilde{R}_{0,T}(w)$ is a piecewise linear function agreeing with w on $(\tau_{-i})_{i\in\mathbb{N}} \cup \{T\}$.

Recall:

$$du = H(Du, x) \cdot dw \ \text{ in } \ \mathbb{R}^d \times (0,T] \qquad u(\cdot, 0) = u_0 \ \text{ in } \ \mathbb{R}^d.$$

Let $S^w(s,t)$ be the nonlinear semigroup of solutions.

Theorem

Let $w \in C_0([0,T])$. Then

$$S^w(0,T) = S^{R_{0,T}(w)}(0,T).$$

Hence

$$\rho_H(w,T) \le L \, \|R_{0,T}(w)\|_{TV([0,T])}.$$

Theorem

Let B be a Brownian motion, $T > 0$. Then, a.s.

$$\left\| R_{0,T}(B) \right\|_{TV([0,T])} < \infty.$$

Sharpness of the upper bound:

Theorem

Let $H(p) = |p|$ on $\mathbb{R}^d$ with $d \geq 1$. Then, for all $T > 0$ and $w \in C_0([0,T])$,

$$\rho_H(w,T) \geq \|\tilde{R}_{0,T}(w)\|_{TV([0,T])}.$$

When $d = 1$, then

$$\rho_H(w,T) = \|\tilde{R}_{0,T}(w)\|_{TV([0,T])}.$$

Proof.

By keeping track of explicit cancellations. □

Sharp speed of propagation in higher dimension?

Consider $H(p) = |p|$ and

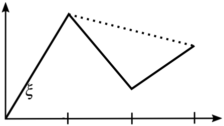
$$\dot{w}(t) = \begin{cases} 4 & \text{for } (0,1) \\ -2 & \text{on } (1,2) \\ 1 & \text{on } (2,3) \end{cases} \quad \text{and } \dot{\tilde{R}}_{0,3}(w) = \begin{cases} 4 & \text{on } (0,1) \\ -\frac{1}{2} & \text{on } (1,3); \end{cases}$$

Which gives

Figure: Evolution of $S^{w}_{|\cdot|}(0,t)$ at $t=0,\ t=1,\ t=2,\ t=3$

Figure: Evolution of $S^{\tilde{R}(w)}_{|\cdot|}(0,t)$ at $t=0,\ t=1,\ t=3$

Let $\delta_1 > \delta_2 > \delta_3 > 0$ and w continuous on $[0,3]$ with



$$\dot{w} = \begin{cases} \delta_1 & \text{on } (0,1), \\ -\delta_2 & \text{on } (1,2), \\ +\delta_3 & \text{on } (2,3). \end{cases}$$

Then

$$\rho_{|\cdot|}(w,3) = \delta_1 + \delta_2 + \delta_3 = \|w\|_{TV([0,3])} = \|R_{0,T}(w)\|_{TV([0,T])} > \|\tilde{R}_{0,T}(w)\|_{TV([0,T])}.$$

Figure: Evolution of $S^{w}_{|.|}(0,\cdot)P_1$

Figure: Evolution of $S^{w}_{|\cdot|}(0,\cdot)P_2$

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