(Analytically) Strong Solutions for Stochastic Partial Differential Equations of Gradient Type

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Introduction and Motivation

- We consider SPDE of the form

$$dX_t = -\partial\varphi(X_t)dt + B_t(X_t)dW_t,$$

where $\varphi : H \to \bar{\mathbb{R}}$ is quasi-convex.
($\exists\ \lambda > 0$ such that $\varphi_\lambda := \varphi + \frac{\lambda}{2}\|\cdot\|_H^2$) is convex)

- Note: $\partial\varphi(x) = \partial\varphi_\lambda(x) - \lambda x$.
- $A := -\partial\varphi$, $\mathcal{D}(A) = \mathcal{D}(\partial\varphi) \subseteq H$

$$dX_t = A(X_t)dt + B_t(X_t)dW_t.$$

Example (Stochastic Porous Medium Equation)

Consider

$$dX_t = \Delta(|X_t|^{p-1}X_t)dt + B_t(X_t)dt, \quad (p \geq 1),$$

on $\mathcal{O} \subseteq \mathbb{R}^d$ open, bounded or $\mathcal{O} = \mathbb{R}^d$ $(d \geq 3)$. With $H = \left(H_0^1(\mathcal{O})\right)^*$,

$$\varphi(v) = \frac{1}{p+1}\|v\|_{L^{p+1}(\mathcal{O})}^{p+1},$$

this is of the form

$$dX_t = -\partial\varphi(X_t)dt + B_t(X_t)dW_t.$$

Strong Solutions

- In general: Solutions to SPDE are (spatially) less regular than to PDE, e.g. $dX_t = AX_t dt + f_t \mathrm{dt}$, then $X_t \in \mathcal{D}(A)$. $dX_t = AX_t dt + dW_t$, then $X_t \in \mathcal{D}(A^{\frac{1}{2}})$.
- Solution: Consider mild or variational solutions
  - mild approach: A generator, $dX_t = (AX_t + F(X_t))dt + B(X_t)dW_t$,

$$X_t = e^{At}X_0 + \int_0^t e^{A(t-s)}F(X_s)ds + \int_0^t e^{A(t-s)}B(X_s)dW_s$$

    with X predictable in H ($X \in \mathcal{D}(F) \cap \mathcal{D}(B)$).
  - variational approach: Gelfand triple $V \subseteq H \subseteq V^*$, extend A to $A : V \to V^*$, variational solution $X \in V$ a.s. (e.g. $\mathcal{D}(A) = \{h \in L^{p+1} |\ |h|^{p-1}h \in H_0^1\}, V = L^{p+1}$).

Strong Solutions

Definition (Strong Solution)

Let $X_0 \in L^2(\Omega, \mathcal{F}_0; H)$. $X \in L^2(\Omega; C([0,T]; H))$ is a strong solution if X is $\mathcal{F}_t$-adapted, continuous in H

- $A(X) = -\partial\varphi(X) \in L^2([0,T] \times \Omega; H)$
- $\mathbb{P}$-a.s.

$$X_t = X_0 - \int_0^t \partial\varphi(X_r)dr + \int_0^t B_r(X_r)dW_r, \ \forall t \in [0,T].$$

If $A = -\partial\varphi$, B sufficiently regular and $X_0 \in \mathcal{D}(\varphi)$ we prove the unique existence of strong solutions.

Regularization (Deterministic Case)

Linear Equations: $dX_t = AX_t dt$

- If A analytic, $X_0 \in H$: $X_t \in \mathcal{D}(A)$ for a.e. $t > 0$ and X_t is a strong solution, i.e.
$$\frac{dX}{dt} = AX_t, \text{a.e.}$$
- Note: If $A = -\partial\varphi$ then A is analytic.

Non-linear Equations: $dX_t = A(X_t)dt$,

- If $A = -\partial\varphi$, $X_0 \in \overline{\mathcal{D}(\varphi)}$ then $t^{\frac{1}{2}}\partial\varphi(X_t) \in L^2([0,T];H)$, $\varphi(X) \in L^1([0,T])$ and
$$\frac{dX}{dt}(t) = -\partial\varphi(X_t), \text{ a.e. } t \in (0,T).$$

Regularization for SPDE

If $A = -\partial\varphi$, B sufficiently smooth, $X_0 \in L^2(\Omega; H)$ then

$$\begin{aligned} \varphi(X) + \|X\|_H^2 &\in L^1([0,T]\times\Omega), \\ t^{\frac{1}{2}}\partial\varphi(X_t) &\in L^2([0,T]\times\Omega; H) \end{aligned}$$

and X is the unique strong solution on each interval $[\delta, T]$, $\delta > 0$ (in particular $X \in \mathcal{D}(A)$ a.s.).

Unified Framework

- Mild approach: Restricted to semilinear equations (e.g. not SPME)
- Variational approach: Not applicable to RDE with high-order growth of the reaction term ($dX_t = (\Delta X_t - X_t^p)dt + B(X_t)dW_t$, $p \geq 1$ arbitrary).
- We present a framework which contains both types of equations (also applicable on not necessarily bounded domains and with various boundary conditions).

Method of Proof

Regularization in Deterministic Case

- Proof based on chain rule for subgradients: Let $X \in W^{1,2}([0,T];H)$, $X_t \in \mathcal{D}(\partial\varphi)$ for a.e. $t \in [0,T]$, then

$$\varphi(X_t) = \varphi(X_0) + \int_0^t (\partial\varphi(X_\tau), \frac{dX}{dt}(\tau))_H d\tau.$$

$$\varphi(X_t) = \varphi(X_0) - \int_0^t \|\partial\varphi(X_\tau)\|_H^2 d\tau.$$

$$\varphi(X_t) = \varphi(X_0) + \int_0^t (\partial\varphi(X_\tau), \frac{dX}{dt}(\tau))_H d\tau.$$

- Proof via Moreau-Yosida approximation φ_λ.
  Note: $\partial\varphi_\lambda = (\partial\varphi)_\lambda \to (\partial\varphi)^0$ on $\mathcal{D}(\partial\varphi)$.
- Problem in the stochastic case:

$$\varphi_\lambda(X_t) = \varphi_\lambda(X_0) + \int_0^t (\partial\varphi_\lambda(X_\tau), dX_\tau)_H$$

$$+ \int_0^t Tr[D^2\varphi_\lambda(X_\tau)B(X_\tau)B(X_\tau)^*]d\tau$$

Standard Galerkin approach

- Let $P_n : H \to H_n$ orthogonal projection onto H_n (w.r.t. $\|\cdot\|_H$).

$$X_t^n = X_0^n - \int_0^t P_n \partial\varphi(X_\tau^n) d\tau + \int_0^t P_n B(X_\tau^n) dW_\tau^n.$$

By Itô's formula:

$$\varphi(X_t^n) = \varphi(X_0^n) + \int_0^t (\partial\varphi(X_\tau^n), dX_\tau^n)_H + \int_0^t Tr[D^2\varphi(X_\tau^n) P_n B(X_\tau^n) P_n B(X_\tau^n)^*] d\tau.$$

- Problem: $\varphi(P_n h) \not\lesssim C\varphi(h)$, i.e. the Galerkin-approximation based on $\|\cdot\|_H$ is not compatible with the "geometry" of φ.

Main Idea

- Galerkin approximation weighted by the "distance" given by φ.
- Recall: $P_n : H \to H_n$ (orth. projection) is the $\|\cdot\|_H$-best-approximation, i.e.

$$\|h - P_n h\|_H = \inf_{g \in H_n} \|h - g\|_H.$$

  Hence, $\|P_n h\|_H \leq 2\|h\|_H$.
- Idea: Consider $\mathcal{P}_n : H \to H_n$ the φ-best-approximation, i.e.

$$\varphi(h - \mathcal{P}_n h) = \inf_{g \in H_n} \varphi(h - g).$$

  Then

$$\varphi(\mathcal{P}_n h) \leq C\varphi(h)$$
$$\varphi(\mathcal{P}_n h) \to \varphi(h).$$

Weighted Galerkin Approximation

- Galerkin approximation:

$$X_t^n = \mathcal{P}_n X_0 - \int_0^t P_n \partial\varphi(X_\tau^n) d\tau + \int_0^t \mathcal{P}_n B(X_\tau^n) dW_\tau^n.$$

- Note: Loose joint monotonicity of $A = -\partial\varphi, B$, i.e.

$$(P^n A(v) - P^n A(w), v - w)_H + \|\mathcal{P}_n B(v) - \mathcal{P}_n B(w)\|_H^2 \not\lesssim C\|v - w\|_H^2.$$

  Solution: Freeze noise (additive case) then use fixed point theorem.

Setup and Main Results

General Setup

Let $V \subseteq H \subseteq V^*$ be a Gelfand triple (e.g. $H = (H_0^1)^*$, $V = L^{p+1}$)

Let U be a separable Hilbert space, $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ a complete, normal, filtered probability space, W_t a cylindrical Brownian motion on U and $B : [0, T] \times \Omega \times H \to L_2(U, H)$ progressively measurable.

Let $\{e_k \in V |\ k \in \mathbb{N}\}$ be an orthonormal basis of H, such that $V_0 = \mathrm{span}\{e_1, \cdots\}$ is dense in V. Define $H_n := \mathrm{span}\{e_1, \cdots, e_n\}$.

Assumptions on the drift

(A1)
- $\varphi : V \to \mathbb{R}_+$ convex, continuous
- subhomogeneous
  (i.e. $\exists\ C > 0$ such that $\varphi(2x) \le C\varphi(x),\ \forall x \in V$),
- bounded level-sets,
- $\varphi(v_n) \to 0$ implies $v_n \to 0$ in V (e.g. $\varphi = \frac{\|\cdot\|_{p+1}^{p+1}}{p+1},\ V = L^{p+1}$).

(A2)
- $\varphi : V \to \mathbb{R}_+$ Gateaux differentiable, $A := -D\varphi : V \to V^*$ hemicontinuous, (e.g. $D\varphi(v)h = \int_{\mathcal{O}} |v|^{p-1} vh\ d\xi$)
- $\varphi \in C^2(H_n)$ $,\forall n \in \mathbb{N}$, with constants $1 = p_0 \le p_2 \le ... \le p_N$

$$\sum_{k=1}^{\infty} D^2\varphi(x)(w_k, w_k) \le C\left(1 + \varphi(x) + \sum_{i=0}^{N}\left(\sum_{k=1}^{\infty}\varphi(w_k)^{\frac{1}{p_i}}\right)^{p_i}\right),$$

for each sequence $w_k \in H_n$, all $x \in H_n$.
(e.g. $D^2\varphi(x)(w_k, w_k) = p\int_{\mathcal{O}} |x|^{p-1}|w_k|^2\ d\xi$).

(A3) (Weak coercivity): For all $v \in V$,

$$2\ {}_{V^*}\langle -D\varphi(v), v\rangle_V \leq C(1 + \|v\|_H^2).$$

(e.g. $2\ {}_{V^*}\langle -D\varphi(v), v\rangle_V = -\int_{\mathcal{O}} |v|^{p+1} = -(p+1)\varphi(v) \leq 0$).

(A4) (Lipschitz noise):

$$\|B_t(v) - B_t(w)\|_{L_2(U,H)}^2 \leq c\|v - w\|_H^2,\ \forall v, w \in V.$$

(A5) (Regularity of the noise): Exists $\tilde{e}_k$ orthonormal basis of U,

$$\begin{aligned}\|B_t(v)\|_{L_{2,\tilde{\varphi}_1,(p_i)}} &:= \sum_{i=0}^{N} \left(\sum_{k=1}^{\infty} \varphi(B_t(v)(\tilde{e_k}))^{\frac{1}{p_i}} \right)^{p_i} \\ &\leq C(f_t + \varphi(v) + \|v\|_H^2),\end{aligned}$$

p_i as in (A2), $f_t \in L^1([0,T] \times \Omega)$ is $\mathcal{F}_t$-adapted.

Theorem (Strong Solution)

Assume (A1)-(A6), $X_0 \in L^2(\Omega; H)$ and $\mathbb{E}(\varphi(X_0)) < \infty$. There is a unique strong solution X with

$$\varphi(X) + \|X\|_H^2 \in L^\infty([0,T]; L^1(\Omega))$$

(recall $\partial\varphi(X) \in L^2([0,T] \times \Omega; H)$).

Definition (Limit Solution)

Let $X_0 \in L^2(\Omega; H)$. $X \in L^2(\Omega; C([0,T]; H))$ is a limit solution if X is $\mathcal{F}_t$-adapted, $X(0) = X_0$ and there exists a sequence X^n of strong solutions such that $X^n \to X$ in $L^2(\Omega; C([0,T]; H))$.

Theorem (Limit Solution)

Assume (A1)-(A6), $X_0 \in L^2(\Omega; H)$. Then there exists a unique limit solution X.

Definition

Let $X_0 \in L^2(\Omega; H)$. $X \in L^2(\Omega; C([0,T]; H))$ is a generalized strong solution if X is $\mathcal{F}_t$-adapted, $X(0) = X_0$, with

- $\partial\varphi(X) \in L^2([\delta, T] \times \Omega; H)$,
- $\mathbb{P}$-a.s.

$$X_t = X_\delta - \int_\delta^t \partial\varphi(X_r)dr + \int_\delta^t B_r(X_r)dW_r, \quad \forall t \in [\delta, T],$$

for all $0 < \delta < T$.

(A4') There exist constants $C_1 > 0$, $C_2 \in \mathbb{R}$ such that

$$2\ {}_{V^*}\langle -D\varphi(v), v\rangle_V \leq C_2(1 + \|v\|_H^2) - C_1\varphi(v),$$

for all $v \in V$. (e.g. $2\ {}_{V^*}\langle -D\varphi(v), v\rangle_V = -(p+1)\varphi(v) \leq 0$).

Theorem

Assume (A1)-(A6), (A4'). Let $X_0 \in L^2(\Omega; H)$ and X be the corresponding limit solution. Then

$$\varphi(X) + \|X\|_H^2 \in L^1([0,T] \times \Omega),$$
$$t^{\frac{1}{2}}\partial\varphi(X_t) \in L^2([0,T] \times \Omega; H)$$

and X is the unique generalized strong solution.

Applications

Stochastic Porous Medium Equation

Consider

$$dX_t = \Delta(|X_t|^{p-1}X_t)dt + B_t(X_t)dW_t,$$

on a bounded domain $\mathcal{O} \subseteq \mathbb{R}^d$ with Dirichlet boundary conditions. Let $H := (H_0^1(\mathcal{O}))^*$, $V = S := L^{p+1}(\mathcal{O})$, and $\varphi(x) := \frac{1}{p+1}\|x\|_{p+1}^{p+1}$.

Theorem (Standard SPME)

Let $X_0 \in L^2(\Omega; H)$, B satisfy (A5), (A6).

- *There exists a unique generalized strong solution X with*

$$\mathbb{E}\int_0^T \|X_t\|_{p+1}^{p+1} + t\||X_t|^{p-1}X_t\|_{H_0^1}^2 dt < \infty.$$

- *The variational solution obtained in [RRW07] coincides with this generalized strong solution.*
- *If additionally $\mathbb{E}\|X_0\|_{p+1}^{p+1} < \infty$, then X is a strong solution with*

$$\sup_{t\in[0,T]} \mathbb{E}\|X_t\|_{p+1}^{p+1} + \mathbb{E}\int_0^T \||X_t|^{p-1}X_t\|_{H_0^1}^2 dt < \infty.$$

Remark (Generalized SPME)

$$dX_t = L\Phi(X_t)dt + B_t(X_t)dW_t,$$

L transient generator.

Stochastic Reaction Diffusion Equation

Consider

$$dX_t = (\Delta X_t + f(X_t))\,dt + B_t(X_t)dW_t,$$

on a bounded domain $\mathcal{O} \subseteq \mathbb{R}^d$ with Dirichlet boundary conditions, f being an polynomial of odd degree N, with negative leading coefficient. Let $H = L^2(\mathcal{O}), S = V = H_0^1(\mathcal{O}) \cap L^{N+1}(\mathcal{O})$ and

$$\varphi(v) := \frac{1}{2}||v||^2_{H_0^1} - \int_{\mathcal{O}} F(v)\ d\xi,$$

where $F' = f$.

Theorem (Standard SRDE)

Let $X_0 \in L^2(\Omega, \mathcal{F}_0; H)$ and B satisfy (A5), (A6).

- *There exists a unique generalized strong solution X with*

$$\mathbb{E}\int_0^T \left(\|X_t\|_{H_0^1}^2 + \|X_t\|_{N+1}^{N+1}\right) + t\left(\|X_t\|_{H^2}^2 + \|X_t\|_{2N}^{2N}\right) dt < \infty.$$

- *If $\mathbb{E}\left(\|X_0\|_{H_0^1}^2 + \|X_0\|_{N+1}^{N+1}\right) < \infty$, then X_t is the unique strong solution with*

$$\sup_{t\in[0,T]} \mathbb{E}\left(\|X_t\|_{H_0^1}^2 + \|X_t\|_{N+1}^{N+1}\right) + \mathbb{E}\int_0^T \|X_t\|_{H^2}^2 + \|X_t\|_{2N}^{2N} dt < \infty.$$

Remark (Generalized SRDE)

$$dX_t = \left(LX_t + \sum_{i=1}^N f_i(X_t)\right) dt + B_t(X_t)dW_t,$$

L non-negative, self-adjoint on $L^2(m)$.