Stochastic scalar conservation laws

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joint work with: Panagiotis E. Souganidis
[G., Souganidis; CMS, 2014], [G., Souganidis; arXiv, 2015].

Outline

Motivation

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- We will consider PDE driven by a 'rough' signal z of the type

$$du + \mathrm{div}(A(x,u) \circ dz) = 0.$$

If A is a diagonal matrix this becomes

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(x,u) \circ dz_j = 0$$

- In this talk, we mainly consider

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(u) \circ d\beta^j = 0.$$

- For example, stochastic Burgers' equation

$$du + \frac{1}{2}\partial_x u^2 \circ d\beta = 0.$$

Motivation

- The motivation comes from two directions: Relation to Hamilton-Jacobi equations, mean-field games.
- Mean-field games going back to Lasry, Lions: Consider the SDE

$$dX_t^i = \sigma\left(X_t^i, \frac{1}{L-1}\sum_{j\neq i}\delta_{X_t^j}\right)\circ d\beta_t \quad \text{in } \mathbb{R}^N$$

  for $i = 1, \dots, L$.
- Then the empirical law of X converges to a measure π_t with density m_t which evolves according to

$$dm + \mathrm{div}(\sigma^*(x,m)\circ d\beta) = 0.$$

  In general σ^* is not a diagonal matrix.

Well-posedness

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- Spatially inhomogeneous case:

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(x,u) \circ d\beta^j = 0.$$

- Pass to the kinetic formulation: Set

$$\chi(t,x,\xi) := 1_{[0,u(t,x)]}(\xi).$$

Well-posedness

- Then (informally)

$$d\chi + \sum_{j=1}^{N} (\partial_u A_j)(x,\xi)\partial_{x_j}\chi\dot{\beta}^j + \left(\sum_{j=1}^{N} \operatorname{div}_{x_j} A(x,\xi)\dot{\beta}^j\right)\partial_\xi \chi = \partial_\xi m.$$

- Test against solutions to

$$d\varphi - \sum_{j=1}^{N} (\partial_u A_j)(x,\xi)\partial_{x_j}\varphi\dot{\beta}^j - \left(\sum_{j=1}^{N} \operatorname{div}_{x_j} A(x,\xi)\dot{\beta}^j\right)\partial_\xi \varphi = 0.$$

Then

$$d\int_{R^{N+1}} \chi\varphi dx d\xi = -\int_{R^{N+1}} \partial_\xi \varphi dm. \qquad (*)$$

- Roughly speaking: u is a pathwise entropy solution if χ satisfies $(*)$.

Theorem (Gess, Souganidis; CMS, 2015)

Pathwise entropy solutions are well-posed.

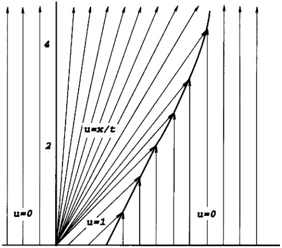
Stochastic Burgers' equation

Stochastic Burgers' equation - a simple example

Stochastic Burgers' equation

- e.g. Burgers' equation

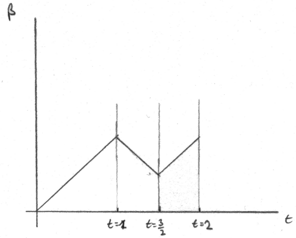
$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0$$
$$u(0) = 1_{[0,1]}$$



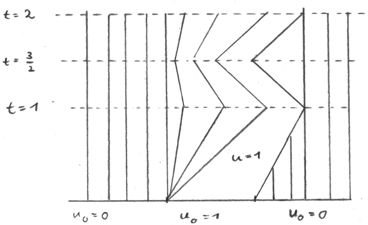
Stochastic Burgers' equation

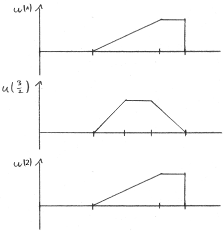
- Inhomogeneous case:

$$\partial_t u + \frac{1}{2}\partial_x u^2 \circ d\beta = 0$$
$$u(0) = 1_{[0,1]}$$



- Solution u:





Long-time behavior

Long-time behavior

Long-time behavior

- We aim to analyze the long-time behavior of

$$\partial_t u + \mathrm{div} A(u) = 0$$

  and

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(u) \circ d\beta_j = 0$$

  on the torus $\mathbb{T}^N$.
- We will show

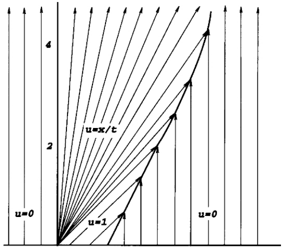
$$u(t) \to \bar{u}_0 = \int_{\mathbb{T}^N} u_0(x)dx \quad \text{for } t \to \infty$$

  in $L^1(\mathbb{T}^N)$.

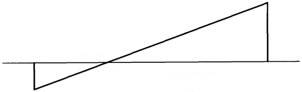
Some results from the deterministic case

- Recall: Burgers' equation

$$du + \frac{1}{2}\partial_x u^2 = 0$$
$$u(0) = 1_{[0,1]}$$



- For large times get asymptotic shape: "N-wave"



Some results from the deterministic case

Some existing results:

- $N = 1$: [Lax; *CPAM*, 1957]. Rate for strictly convex flux:

$$\|u(t) - \bar{u}_0\|_\infty \lesssim t^{-1}.$$

- $N = 2$: [Engquist, E; *CPAM*; 1993]. Rate for strictly convex flux:

$$\|u(t) - \bar{u}_0\|_\infty \lesssim t^{-1}.$$

- $N \geq 1$: [Chen, Frid; *ARMA*; 1999], [Chen, Perthame; *Proc. AMS*; 2009]. If A is 'genuinely nonlinear' then

$$\|u(t) - \bar{u}_0\|_1 \to 0 \quad \text{for } t \to \infty.$$

- Open problem: Rate of convergence for $N \geq 3$?

(New) rates for the deterministic case

- Assume that the flux A is *genuinely nonlinear*: there exist $\theta \in (0,1]$ and $C > 0$ such that, for all $\sigma \in S^{N-1}$, $z \in \mathbb{R}$ and $\varepsilon > 0$,

$$|\{\xi \in \mathbb{R} : |A'(\xi) \cdot \sigma - z| \leq \varepsilon\}| \leq C\varepsilon^{\theta}.$$

  For example: For A strictly convex, $N = 1$ we have $\theta = 1$.

- Let u be the unique entropy solution to

$$\partial_t u + \sum_{j=1}^{N} \partial_{x_j} A_j(u) = 0.$$

Theorem (G., Souganidis; 2015)

For $t \geq 1$ and $u_0 \in L^2(\mathbb{T}^N)$,

$$\|u(\cdot, t; \cdot, u_0) - \bar{u}_0\|_1 \leq t^{-\frac{\theta}{2+\theta}} \left(\|u_0\|_2^2 + 1\right).$$

(New) rates for the stochastic case

- Let us return to

$$du + \sum_{j=1}^{N} \partial_{x_j} A_j(u) \circ d\beta_j = 0.$$

- Again assume that A is genuinely nonlinear.

Theorem (G., Souganidis; 2015)

For $t \geq 1$ and $u_0 \in L^2(\mathbb{T}^N)$,

$$\mathbb{E}\|u(\cdot,t;\cdot,u_0) - \bar{u}_0\|_1 \leq \left(\|u_0\|_2^2 + 1\right) t^{-\frac{\theta}{3+\theta}}.$$

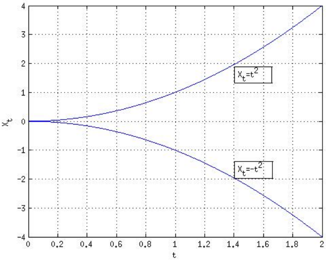
- E.g. $\theta = 1$: deterministic rate $t^{-\frac{1}{3}}$, stochastic rate $t^{-\frac{1}{4}}$. But: No claim of optimality.
- Note: Brownian motion scales like $\sqrt{t}$, which "slows down" characteristics.

Regularization by noise

Regularization by noise

Regularization by noise - linear case

- We recall: Consider
$$du + b(x) \cdot \nabla u = 0 \qquad \text{(TE)}$$
for non-Lipschitz b (but, say, Hölder continuous). E.g. $b(x) = \operatorname{sgn}(x)\sqrt{|x|}$.
- In general, characteristics collide causing shocks (i.e. discontinuities). Solution is not better than $u(t) \in BV$ even if u_0 is smooth.
- Weak solutions are non-unique: e.g. $b(x) = \operatorname{sgn}(x)\sqrt{|x|}$

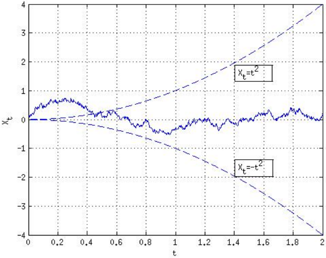


- Question: Can noise restore uniqueness or increase regularity?

Regularization by noise - linear case

- Consider
$$du + b(x)\cdot\nabla u = -\nabla u \circ dW_t. \qquad \text{(STE)}$$
- Two (related) effects: regularization by noise, well-posedness by noise.
- Regularization by noise [Flandoli, Fedrizzi; *JFA*, 2013]: If u_0 is smooth then $u(t)$ is smooth.
- Well-posedness by noise [Flandoli, Gubinelli, Priola; *Invent. Math.*, 2010]: Weak solutions to (STE) are unique.

$$du + b(x)\cdot\nabla u = -\nabla u \circ dW_t$$
$$b(x) = \operatorname{sgn}(x)\sqrt{|x|}$$



- Entirely open: What about the nonlinear case, e.g. Burgers?

Quasi-solutions and averaging

- Consider the Burgers' equation

$$\partial_t u + \partial_x u^2 = 0 \quad \text{on } \mathbb{T} \tag{B}$$

- Weak solutions to (B) are not unique.
- We consider quasi-solutions [De Lellis, Otto, Westdickenberg; *ARMA*, 2003]: A weak solution u to (B) is a quasi-solution, if for some Radon measure m

$$\partial_t \chi + \xi \partial_x \chi = \partial_\xi m.$$

  Quasi-solutions to (B) are not unique.
- [De Lellis, Westdickenberg; *AIHP*, 2003]: There is a quasi-solution to (B) such that

$$u(t) \notin W^{1,\lambda} \qquad \text{for all } \lambda > \frac{1}{3}.$$

- Question: Does noise improve the situation?

(New) results for the stochastic case

- Consider the stochastic Burgers' equation

$$du + \partial_x u^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T} \tag{SB}$$

Theorem (G., Souganidis; 2015)

Let u be a pathwise quasi-solution to (SB). Then, $t > 0$,

$$u(t) \in W^{\lambda,1} \quad \textit{for all } \lambda \in (0, \frac{1}{2}), \mathbb{P}\textit{-a.s.}.$$

- Thus: quasi-solutions to (SB) are more regular than to (B), i.e. regularization by noise.

Thanks

Thanks!