

Finite time extinction for stochastic sign fast diffusion and self-organized criticality.

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Outline

- 1 Introduction and main result
- 2 Finite time extinction for deterministic BTW
- 3 Finite time extinction for stochastic BTW

The stochastic BTW model

- In [Díaz-Guilera; EPL (Europhysics Letters), 1994], [Giacometti, Diaz-Guilera; Phys. Rev. E, 1998], [Díaz-Guilera; Phys. Rev. A, 1992] the following model for self-organized criticality (SOC) was introduced:

$$dX_t = \Delta H(X_t - X^c)dt + B(X_t - X^c)dW_t,$$

with appropriate diffusion coefficients B and Dirichlet boundary conditions.

- We study linear multiplicative noise, i.e.

$$dX_t = \Delta H(X_t - X^c)dt + \sum_{k=1}^N f_k(X_t - X^c)d\beta_t^k. \quad (\text{SOC})$$

with zero Dirichlet boundary conditions.

- Question: Does the diffusion end in finite time?

Finite time extinction and SOC

- We will restrict to the supercritical case, i.e. supposing $x_0 \geq X^c$.
- Substituting $X \rightarrow X - X^c$ and using $X \geq 0$ yields

$$dX_t = \Delta \operatorname{sgn}(X_t) dt + \sum_{k=1}^N f_k X_t d\beta_t^k,$$

$$X(0) = x_0$$

with $x_0 \geq 0$ and zero Dirichlet boundary conditions:

$$\operatorname{sgn}(X(t, \xi)) = 0, \quad \text{on } \partial\mathcal{O}.$$

- Informally:

$$\Delta \operatorname{sgn}(X) = \delta_0(X) \Delta X + \operatorname{sgn}''(X) |\nabla X|^2.$$

- Diffusion ends in finite time = Finite time extinction.

The stochastic BTW model

- Recall:

$$dX_t = \Delta \operatorname{sgn}(X_t) dt + \sum_{k=1}^N f_k X_t d\beta_t^k, \quad (\text{BTW})$$

with zero Dirichlet boundary conditions.

- Finite time extinction can be reformulated in terms of the extinction time

$$\tau_0(\omega) := \inf\{t \geq 0 \mid X_t(\omega) = 0, \text{ a.e. in } \mathcal{O}\}.$$

We distinguish the following concepts:

(F1) Extinction with positive probability for small initial conditions:

$$\mathbb{P}[\tau_0 < \infty] > 0, \text{ for small } X_0 = x_0.$$

(F2) Finite time extinction: $\mathbb{P}[\tau_0 < \infty] = 1$, for all $X_0 = x_0$.

- (F2) for (BTW) has been addressed but left open in: [V. Barbu, MMAS, 2013], [M. Röckner, F-Y. Wang, JLMS, 2013], [V. Barbu, G. Da Prato, M. Röckner, JMAA, 2012], [V. Barbu, M. Röckner, CMP, 2012], [V. Barbu, G. Da Prato, M. Röckner, CMP, 2009], [V. Barbu, G. Da Prato, M. Röckner, CRMAS, 2009]

Main result

Theorem (Main result)

Let $x_0 \in L^\infty(\mathcal{O})$, X be the unique variational solution to BTW and let

$$\tau_0(\omega) := \inf \{ t \geq 0 \mid X_t(\omega) = 0, \text{ for a.e. } \xi \in \mathcal{O} \}.$$

Then finite time extinction holds, i.e.

$$\mathbb{P}[\tau_0 < \infty] = 1.$$

For every $p > \frac{d}{2} \vee 1$, the extinction time $\tau_0(\omega)$ may be chosen uniformly for x_0 bounded in $L^p(\mathcal{O})$.

Finite time extinction for deterministic PDE

Finite time extinction for deterministic PDE

Finite time extinction for singular ODE

- Consider the singular ODE

$$\dot{f} = -cf^\alpha, \quad \alpha \in (0,1), \quad c > 0.$$

- Then:

$$(f^{1-\alpha})' = -(1-\alpha).$$

- We obtain

$$f^{1-\alpha}(t) = f^{1-\alpha}(0) - (1-\alpha)ct$$

which implies finite time extinction.

Finite time extinction and SOC

- [Diaz, Diaz; CPDE, 1979] finite time extinction (FTE) was first proven for

$$\frac{\partial}{\partial t} X(t, \xi) = \Delta \operatorname{sgn}(X(t, \xi)).$$

- In [Barbu; MMAS, 2012] another (more robust) approach based on energy methods was introduced.

Finite time extinction and SOC

- Informally the proof boils down to a combination of an L^1 and an L^∞ estimate of the solution:
- Informal L^∞ estimate:

$$\|X(t)\|_\infty \leq \|x_0\|_\infty, \quad \forall t \geq 0.$$

- Informal L^1 -estimate:

$$\begin{aligned} \partial_t \int_{\mathcal{O}} |X(t, \xi)| d\xi &= \int_{\mathcal{O}} \operatorname{sgn}(X(t, \xi)) \Delta \operatorname{sgn}(X(t, \xi)) d\xi \\ &= - \int_{\mathcal{O}} |\nabla \operatorname{sgn}(X(t, \xi))|^2 d\xi \\ &\leq - \left(\int_{\mathcal{O}} |\operatorname{sgn}(X(t, \xi))|^p d\xi \right)^{\frac{2}{p}} \\ &\leq - (|\{\xi \mid X(t, \xi) \neq 0\}|)^{\frac{2}{p}}, \end{aligned}$$

for some (dimension dependent) $p > 2$. Note: $\frac{2}{p} < 1$.

Finite time extinction and SOC

- Observe

$$\begin{aligned} \int_{\mathcal{O}} |X(t, \xi)| d\xi &\leq \|X(t)\|_{\infty} |\{\xi | X(t, \xi) \neq 0\}|. \\ &\leq \|x_0\|_{\infty} |\{\xi | X(t, \xi) \neq 0\}|. \end{aligned}$$

- Using this above gives

$$\partial_t \int_{\mathcal{O}} |X(t, \xi)| d\xi \leq -\frac{1}{\|x_0\|_{\infty}^{\frac{2}{p}}} \left(\int_{\mathcal{O}} |X(t, \xi)| d\xi \right)^{\frac{2}{p}}.$$

- We are left with the singular ODE

$$\dot{f} = -cf^{\alpha}, \quad \alpha \in (0, 1), \quad c > 0$$

for which we have seen that finite time extinction holds.

Finite time extinction for stochastic BTW

Finite time extinction for stochastic BTW

Transformation

- Recall:

$$dX_t = \Delta \operatorname{sgn}(X_t) dt + \sum_{k=1}^N f_k X_t d\beta_t^k,$$

- Our approach to FTE will be based on considering the following transformation: Set $\mu_t := \sum_{k=1}^N f_k \beta_t^k$, $\tilde{\mu} := \sum_{k=1}^N f_k^2$ and $Y_t := e^{-\mu_t} X_t$. An informal calculation shows

$$\partial Y_t \in e^{\mu_t} \Delta \operatorname{sgn}(Y_t) - \tilde{\mu} Y_t. \quad (*)$$

- Compare the deterministic setting:

$$\partial Y_t \in \Delta \operatorname{sgn}(Y_t).$$

Outline of the proof

- There are two main ingredients of the proof:
 - ① A uniform control on $\|X_t\|_p$ for all $p \geq 1$.
 - ② An energy inequality for a weighted L^1 -norm.
- On an intuitive level the arguments become clear by approximating

$$r^{[m]} := |r|^{m-1}r \rightarrow \text{sgn}, \quad \text{for } m \downarrow 0.$$

To make these arguments rigorous, in fact a different (non-singular, non-degenerate) approximation of sgn is used.

- In the following let Y_t be a solution to

$$\partial_t Y_t \in e^{\mu t} \Delta Y_t^{[m]} - \tilde{\mu} Y_t.$$

Step 1: Informal L^p bound

- **Step 1:** A uniform control on $\|X_t\|_p$ for all $p \geq 1$.
- We may informally compute for all $p \geq 1$:

$$\begin{aligned} \partial_t \int_{\mathcal{O}} |Y_t|^p d\xi &= p \int_{\mathcal{O}} Y_t^{[p-1]} e^{\mu t} \Delta Y_t^{[m]} d\xi \\ &= -\frac{4(p-1)mp}{(p+m-1)^2} \int_{\mathcal{O}} e^{\mu t} \left(\nabla |Y_t|^{\frac{p+m-1}{2}} \right)^2 d\xi \\ &\quad + \frac{pm}{p+m-1} \int_{\mathcal{O}} |Y_t|^{p+m-1} \Delta e^{\mu t} d\xi. \end{aligned}$$

- Taking $p > 1$ and then $m \rightarrow 0$ we may “deduce” from this

$$\partial_t \int_{\mathcal{O}} |Y_t|^p d\xi \leq 0.$$

Step 2: Informal “ L^1 ” bound

- **Step 2:** An energy inequality for a weighted L^1 -norm.

$$\begin{aligned} \partial_t \int_{\mathcal{O}} |Y_t|^p d\xi &= -\frac{4(p-1)mp}{(p+m-1)^2} \int_{\mathcal{O}} e^{\mu t} \left(\nabla |Y_t|^{\frac{p+m-1}{2}} \right)^2 d\xi \\ &\quad + \frac{pm}{p+m-1} \int_{\mathcal{O}} |Y_t|^{p+m-1} \Delta e^{\mu t} d\xi, \quad p \geq 1. \end{aligned}$$

- Choose $p = m+1$ and let $m \rightarrow 0$. We obtain

$$\partial_t \int_{\mathcal{O}} |Y_t| d\xi = - \int_{\mathcal{O}} e^{\mu t} (\nabla \operatorname{sgn}(Y_t))^2 d\xi + \frac{1}{2} \int_{\mathcal{O}} \Delta e^{\mu t} d\xi$$

- Recall: deterministic case

$$\partial_t \int_{\mathcal{O}} |Y_t| d\xi = - \int_{\mathcal{O}} |\nabla \operatorname{sgn}(Y_t)|^2 d\xi.$$

Step 2: Informal “ L^1 ” bound

Key trick: Use a weighted L^1 -norm

- Let φ be the classical solution to

$$\begin{aligned}\Delta\varphi &= -1, & \text{on } \mathcal{O} \\ \varphi &= 1, & \text{on } \partial\mathcal{O}.\end{aligned}$$

Note $1 \leq \varphi \leq \|\varphi\|_\infty =: C_\varphi$.

- We informally compute

$$\partial_t \int_{\mathcal{O}} \varphi |Y_t| d\xi = - \int_{\mathcal{O}} \varphi e^{\mu t} (\nabla \text{sgn}(Y_t))^2 d\xi + \frac{1}{2} \int_{\mathcal{O}} \Delta(\varphi e^{\mu t}) d\xi.$$

- Note

$$\Delta(\varphi e^{\mu t}) = -e^{\mu t} + 2\nabla\varphi \cdot \nabla e^{\mu t} + \varphi \Delta e^{\mu t}$$

has a negative sign for small times ($e^{\mu t} \approx 1$)!

- Shift the initial time

$$\partial_t \int_{\mathcal{O}} e^{-\mu s} \varphi |Y_t| d\xi = - \int_{\mathcal{O}} e^{\mu t - \mu s} \varphi (\nabla \text{sgn}(Y_t))^2 d\xi + \frac{1}{2} \int_{\mathcal{O}} \text{sgn}(Y_t)^2 \Delta e^{\mu t - \mu s} \varphi d\xi$$

Thanks

Thanks!