Fluctuations in conservative systems and SPDEs

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joint work with Ben Fehrman [https://arxiv.org/abs/1910.11860]

and Daniel Heydecker [https://arxiv.org/abs/2303.11289]

slides online: bgess.de → talks

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Content

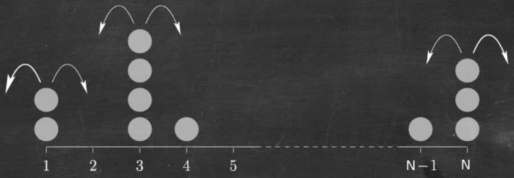
From interacting particle systems to conservative SPDEs

From large deviations to parabolic-hyperbolic PDE with irregular drift

Parabolic-hyperbolic PDE with irregular drift

From interacting particle systems to conservative SPDEs

The zero range process (could also consider simple exclusion, independent particles, ..).



- State space $\mathbb{M}_N := \mathbb{N}_0^{\mathbb{T}_N}$, i.e. configurations $\eta : \mathbb{T}_N \to \mathbb{N}_0$: System in state η if container k contains $\eta(k)$ particles.
- Local jump rate function $g : \mathbb{N}_0 \to \mathbb{R}_0^+$.
- Translation invariant, asymmetric, zero mean transition probability

$$p(k, l) = p(k - l), \quad \sum_k kp(k) = 0.$$

- Markov jump process $\eta(t)$ on $\mathbb{M}_N$.
- $\eta(k, t) =$ number of particles in box k at time t.

- **Hydrodynamic limit?** Multi-scale dynamics

- Empirical density field: $\mu^N(x,t) := \frac{1}{N}\sum_k \delta_{\frac{k}{N}}(x)\eta(k,tN^2)$.
- [Hydrodynamic limit - Ferrari, Presutti, Vares; 1987]

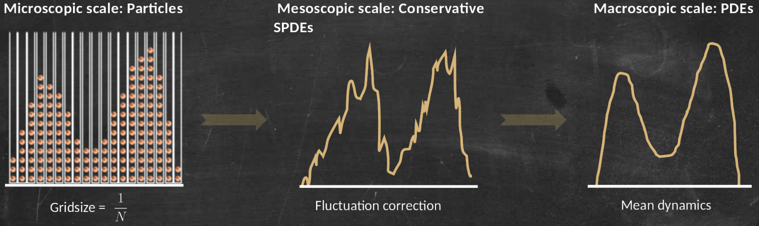
$$\mu^N(t) \rightharpoonup^* \bar{\rho}(t)\,dx$$

with

$$\partial_t \bar{\rho} = \partial_{xx}\Phi(\bar{\rho})$$

with Φ the mean local jump rate $\Phi(\rho) = \mathbb{E}_{\nu_\rho}[g(\eta(0))]$.
- Loss of information:
  - ▶ Fluctuations, rare events / large deviations?
  - ▶ Model / Approximation error: $\mu^N = \bar{\rho} + O(N^{-\frac{1}{2}})$.

Fluctuating Hydrodynamics?



Ansatz: Conservative SPDEs

$$\partial_t \rho^N = \partial_{xx}\Phi(\rho^N) + N^{-\frac{1}{2}}\partial_x\left(\Phi^{\frac{1}{2}}(\rho^N)\xi^N\right),$$

with ξ^N noise, spatially correlated with decorrelation length $\frac{1}{N}$, and white in time, e.g. $\xi^N(t,x) = \sum^N e_k(x)\dot{\beta}^k_t$.

Informally, correct large deviations:

- Recall
$$\partial_t \rho^N = \partial_{xx}\left(\Phi(\rho^N)\right) + N^{-\frac{1}{2}}\partial_x\left(\Phi^{\frac{1}{2}}(\rho^N)\xi^N\right).$$

- Rare events: (Im-)probability to observe a fluctuation ρ:
$$\mathbb{P}[\rho^N \approx \rho] = e^{-N\,I(\rho)} \quad N \text{ large}$$

- Informally applying the contraction principle to the solution map
$$F : N^{-\frac{1}{2}}\xi \mapsto \rho$$
yields as a rate function
$$I(\rho) = \inf\{I_\xi(g) : F(g) = \rho\}.$$

- Schilder's theorem for Brownian sheet suggests
$$I_\xi(g) = \int_0^T \int_{\mathbb{T}} |g|^2\, dxdt.$$

- Get
$$I(\rho) = \inf\left\{\int_0^T \int_{\mathbb{T}} |g|^2\, dxdt : \partial_t\rho = \partial_{xx}\left(\Phi(\rho)\right) + \partial_x\left(\Phi^{\frac{1}{2}}(\rho)g\right)\right\}.$$

Model / Approximation error:

$$\partial_t \rho^N = \partial_{xx}\Phi(\rho^N) + \partial_x \left(\Phi^{\frac{1}{2}}(\rho^N) N^{-\frac{1}{2}} \xi^N\right).$$

Central limit theorems predict

$$\rho^N = \bar{\rho} + N^{-\frac{1}{2}} Y^1 + O(N^{-1})$$
$$\mu^N = \bar{\rho} + N^{-\frac{1}{2}} Y^1 + O(N^{-1}).$$

Conclude: Higher order of approximation

$$\mu^N = \rho^N + O(N^{-1}).$$

Challenges:

- Well-posedness of conservative SPDEs (2013–): [Lions, Perthame, Souganidis; 2013, 2014], [G., Souganidis; 2015, 2017], [Fehrman, G.; 2021], [Dareiotis, G.; 2020], [Fehrman, G.; 2022].
- Large deviations: [Fehrman, G.; 2022], [Mariani, 2010]
- Expansions / quantified central limit theorems: [Dirr, Fehrman, G.; 2021], Linear case [Cornalba, Fischer, Ingmanns, Raithel]; [Djurdjevac, Kremp, Perkowski].

From large deviations to parabolic-hyperbolic PDE with irregular drift

Rare events: (Im-)probability to observe a fluctuation ρ:

$$\mathbb{P}[\mu^N \approx \rho] = e^{-N\, I(\rho)} \quad N \text{ large}$$

A bit more precisely, for every open set O,

$$\mathbb{P}[\mu^N \in \bar{O}] \lesssim e^{-N \inf_{\rho \in \bar{O}} I(\rho)}$$

$$e^{-N \inf_{\rho \in O} I(\rho)} \lesssim \mathbb{P}[\mu^N \in O]$$

Zero range process

$$I(\rho) = \inf\{ \underbrace{\int_{t,x} |\partial_x H|^2 \Phi(\rho)}_{=:\|H\|_{H^1_{\Phi(\rho)}}} : \quad \underbrace{\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x H)}_{\text{"controlled nonlinear Fokker-Planck equation"}} \quad \}.$$

Theorem ([Large deviation principle, Kipnis, Olla, Varadhan; 1989 & Benois, Kipnis, Landim; 1995])

For every open set $O \subseteq D([0,T],\mathcal{M}_+)$ we have

$$\mathbb{P}[\mu^N \in \bar{O}] \lesssim e^{-N \inf_{\rho\in\bar{O}} I(\rho)}$$

$$\mathbb{P}[\mu^N \in \bar{O}] \lesssim e^{-N \inf_{\rho\in\bar{O}} I(\rho)}$$

$$e^{-N \inf_{\rho\in O} J(\rho)} \lesssim \mathbb{P}[\mu^N \in O]$$

where $J = \overline{I_{|A}}$ and A is the set of nice fluctuations $\mu = \rho\, dx$ with ρ a solution to

$$\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x H)$$

for some $\underline{H \in C^{1,3}_{t,x}}$.

This is a frequently observed problem: E.g. Fluctuations around Boltzmann equation [Rezakhanlou 1998], [Bodineau, Gallagher, Saint-Raymond, Simonella 2020].
Counter-examples for Boltzmann [Heydecker; 2021].

Problem:

$$I = J = \overline{I_{|A}}\,?$$

Existence of a "recovery sequence"? Given fluctuation ρ so that $I(\rho dx) < \infty$, i.e. for some $H \in H^1_{\Phi(\rho)}$,

$$\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x \underbrace{H}_{\in H^1_{\Phi(\rho)}}).$$

Need to find sequence of nice fluctuations $\rho^\varepsilon \in A$ so that $\rho^\varepsilon \to \rho$ and $I(\rho^\varepsilon) \to I(\rho)$. That is, find $H^\varepsilon \in C^{1,3}([0,T] \times \mathbb{T})$ so that

$$I(\rho^\varepsilon) = \|H^\varepsilon\|_{L^2_t H^1_{\Phi(\rho)}} \to \|H\|_{L^2_t H^1_{\Phi(\rho)}} = I(\rho)$$

and

$$\partial_t \rho^\varepsilon = \partial_{xx}\Phi(\rho^\varepsilon) + \partial_x(\Phi(\rho^\varepsilon)\partial_x H^\varepsilon)$$

satisfies $\rho^\varepsilon \to \rho$.

Difficult: Open problem for the zero range process since [Benois, Kipnis, Landim; 1995].

Recall: Informally the LDP expected from

$$\partial_t \rho = \partial_{xx}\Phi(\rho) + N^{-\frac{1}{2}}\partial_x(\Phi^{\frac{1}{2}}(\rho)\xi^N)$$

is

$$I(\rho) = \inf\{\int_0^T \int_{\mathbb{T}} |g|^2 \, dxdt \ : \underbrace{\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)g)}_{\text{"skeleton equation"}}\}.$$

Observation: Stability properties are better studied via the skeleton PDE

$$\begin{aligned}\partial_t \rho &= \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x H) \\ &= \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)\underbrace{\Phi^{\frac{1}{2}}(\rho)\partial_x H}_{L^2_{t,x}}) \\ &= \underbrace{\partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)\overbrace{g}^{\in L^2_{t,x}})}_{\text{"skeleton equation"}}\end{aligned}$$

Stability: $g \mapsto \rho$, $L^2_{t,x} \to L^1_{t,x}$ continuous?

I.e. Stability and uniqueness of a PDE with irregular coefficients $g \in L^2_{t,x}$.

Eventually it turns out that stability is studied better on the level of the skeleton equation than on the level of the nonlinear Fokker-Planck equation.

Parabolic-hyperbolic PDE with irregular drift

Skeleton equation

$$\partial_t \rho = \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho) \underbrace{g}_{\in L^2_{t,x}}).$$

How difficult is the well-posedness?

- Difficulty: Stable a-priori bound? L^p framework does not work.
- Do we expect non-concentration of mass / well-posedness?

Scaling and criticality of the skeleton equation

- We consider, $\Phi(\rho) = \rho^m$,

$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g)$$

  with $g \in L^q_t L^p_x$ and $\rho_0 \in L^r_x$.
- Via rescaling ("zooming in"):
  - $p = q = 2$ is critical.
  - $r = 1$ is critical, $r > 1$ is supercritical.

Recall: [Le Bris, Lions; CPDE 2008], [Karlssen, Risebro, Ohlberger, Chen, ...]

$$\partial_t \rho = \frac{1}{2} D^2 : (\sigma\sigma^* \rho) + \mathrm{div}(\rho g)$$

needs $g \in W^{1,1}_{loc,x}$, $\mathrm{div}\, g \in L^\infty$.

Overview of ingredients of the proof:

- **Part 1:** Apriori-bounds; entropy-entropy dissipation estimates
- **Part 2:** Extending the concepts of DiPerna-Lions, Ambrosio, Le Bris-Lions to nonlinear PDE (but going beyond).
- **Part 3:** Uniqueness for renormalized entropy solutions (variable doubling): New treatment of kinetic dissipation measure. Exploit finite *singular* moments.

Part 1: Apriori-bounds

- Consider
$$\partial_t \rho = \Delta \rho^m + \mathrm{div}(\rho^{\frac{m}{2}} g) \quad \text{on } \mathbb{R}_+ \times \mathbb{T}^d \tag{*}$$
with $g \in L^2_{t,x}$, $m \in [1,\infty)$. E.g.
$$\partial_t \rho = \Delta \rho + \mathrm{div}(\rho^{\frac{1}{2}} g).$$
- Use entropy-entropy dissipation: Evolution of entropy given by $\int_{\mathbb{T}^d} \log(\rho)\rho$. Informally gives
$$\int_x \log(\rho)\rho\,\Big|_0^t + \int_0^t \int_x (\nabla \rho^{\frac{m}{2}})^2 \lesssim \int_0^t \int_x g^2.$$
- Caution: Can only be true for non-negative solutions.
- Non-standard weak solutions, rewriting $(*)$ as
$$\partial_t \rho = 2\mathrm{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \mathrm{div}(\rho^{\frac{m}{2}} g).$$
- Stability in the control: for $g^\varepsilon \rightharpoonup g$ in $L^2_{t,x}$ by compactness $\rho^\varepsilon \to \hat{\rho}$ weak solution to $(*)$.
- Conclusion: Have to prove uniqueness within this class of solutions.

Part 2: Renormalization

Recall: Linear case [DiPerna, Lions, Invent. 1989; Ambrosio Invent. 2004]

$$\partial_t \rho = \mathrm{div}(\rho g).$$

Then ρ is a renormalized solution, if for all smooth f we have

$$\partial_t f(\rho) = \mathrm{div}(f(\rho) g) - (f(\rho) + f'(\rho)\rho)\mathrm{div} g.$$

Let ρ be a weak solution to

$$\partial_t \rho = 2\mathrm{div}(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + \mathrm{div}(\rho^{\frac{m}{2}} g).$$

Show that every weak solution is a renormalized (= kinetic) solution (merging renormalization [DiPerna, Lions; Ambrosio] with kinetic solutions [Lions, Perthame, Tadmor, J. Amer. Math. Soc. 1994]).

Let

$$\chi(t,x,\xi) = f_\xi(\rho(x,t)) = 1_{0<\xi<\rho(x,t)} - 1_{\rho(x,t)<\xi<0}.$$

Then, informally,

$$\partial_t \chi = m\xi^{m-1}\Delta_x \chi - g(x,t)(\partial_\xi \xi^{\frac{m}{2}})\nabla_x \chi + (\nabla_x g(x,t))\xi^{\frac{m}{2}}\partial_\xi \chi + \partial_\xi q$$

with p parabolic defect measure

$$q = \delta(\xi - \rho(x,t))|\nabla \rho^{\frac{m+1}{2}}|^2.$$

- Note: Additional commutator errors by commuting convolution and nonlinearities.
- Commutator estimate using non-standard (optimal) regularity $\rho^{\frac{m}{2}} \in L_t^2 \dot{H}_x^1$
- Additional renormalization step to compensate low time integrability $\rho^{\frac{m}{2}} g \in L_t^1 L_x^1$.

Theorem

A function $\rho \in L_t^\infty L_x^1$ is a weak solution to

$$\partial_t \rho = 2 div(\rho^{\frac{m}{2}} \nabla \rho^{\frac{m}{2}}) + div(\rho^{\frac{m}{2}} g)$$

if and only if ρ is a renormalized entropy solution (kinetic solution).

Part 3: Uniqueness for renormalized entropy solutions (variable doubling)

- Established arguments [Chen, Perthame; 2003] not applicable.
- Additional errors from space-inhomogeneity (with little regularity)
- Note: Entropy dissipation measure

$$q(x,\xi,t) = \delta(\xi - \rho(x,t))|\nabla\rho^{\frac{m+1}{2}}|^2 = \delta(\xi - \rho(x,t))\frac{\xi^m}{\xi^{m-1}}|\nabla\rho^{\frac{m}{2}}|^2$$

  does not satisfy

$$\lim_{|\xi|\to\infty}\int_{t,x} q(x,\xi,t)\,dxdt = 0.$$

- Only finite singular moment

$$\int_{t,x,\xi}|\xi|^{-1}q(x,\xi,t)\,d\xi dxdt < \infty.$$

Theorem (The skeleton equation, Fehrman, G. 2022)

Let $g \in L^2_{t,x}$, ρ_0 non-negative and $\int \rho_0 \log(\rho_0) dx < \infty$. There is a unique weak solution to

$$\partial_t \rho = \Delta \Phi(\rho) + div(\Phi^{\frac{1}{2}}(\rho) g).$$

The map $g \mapsto \rho$, $L^2_{t,x} \to L^1_{t,x}$, is weak-strong continuous. E.g. including all $\Phi(\rho) = \rho^m$, $m \in [1, \infty)$.

Theorem (LDP for zero range process, G., Heydecker, 2023)

The rescaled zero range process satisfies the full large deviations principle with rate function

$$I(\rho) = \|\partial_t \rho - \partial_{xx} \Phi(\rho)\|_{H^{-1}_{\Phi(\rho)}}.$$

References

B. Fehrman and B. Gess.
Non-equilibrium large deviations and parabolic-hyperbolic PDE with irregular drift.
arXiv:1910.11860 [math], Mar. 2022.

B. Gess and D. Heydecker.
A Rescaled Zero-Range Process for the Porous Medium Equation: Hydrodynamic Limit, Large Deviations and Gradient Flow, Mar. 2023.

Advertisement: Two open PostDoc positions at Bielefeld University (CRC 1283, and ERC CoG "FluCo") in stochastic analysis, in particular,

- stochastic PDEs
- non-equilibrium statistical mechanics
- mathematics of machine learning
- stochastic dynamics.

Construction of the recovery sequence ρ^n with smooth controls H^n: Recall

$$\begin{aligned}\partial_t\rho &= \partial_{xx}\Phi(\rho) + \partial_x(\Phi(\rho)\partial_x H)\\ &= \partial_{xx}\Phi(\rho) + \partial_x(\Phi^{\frac{1}{2}}(\rho)\underbrace{\Phi^{\frac{1}{2}}(\rho)\partial_x H}_{g}).\end{aligned}$$

Let

$$g_n = g * \kappa^{\frac{1}{n}},\ \rho_{0,n} = ((\rho_0 \vee \frac{1}{n}) \wedge n) * \kappa^{\frac{1}{n}}$$

$$\psi_n = 0 \text{ on } [0, \frac{1}{2n}] \cup [2n, \infty) \ \ \psi_n = 1 \text{ on } [\frac{1}{n}, n]$$

Let ρ_n be the solution to

$$\partial_t\rho_n = \Delta\Phi(\rho_n) - \nabla\cdot(\Phi^{\frac{1}{2}}(\rho_n)\psi(\rho_n)g_n).$$

By comparison get

$$1/2n \leq \rho_n \leq 2n,$$

by parabolic regularity get ρ^n smooth. Consider the equation

$$\partial_t\rho_n = \Delta\Phi(\rho_n) - \nabla\cdot(\Phi(\rho_n)\nabla H_n)$$

as a (non-degenerate) elliptic equation for H_n. We get existence of a smooth H_n and thus ρ^n is a nice fluctuation.

The entropy dissipation estimate for ρ_n is still applicable since $\|\psi(\rho_n)g_n\|_{L^2} \leq \|g\|_{L^2}$. This allows to show convergence to a solution ρ, which by uniqueness is the pre-given function ρ.